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ASYMMETRIC DOUBLE WEIBULL DISTRIBUTION AND ITS APPLICATIONS

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Summary

The dissertation examines the asymmetric Weibull distributions extended to the higher dimensional setting. Newly developed definitions along with the methods of estimation of the parameters are presented. This provides the basis for applications in modelling financial data including the univariate and the multivariate double Weibull model for currency exchange rates. The dissertation starts with an extensive review of the univariate Weibull distribution followed by its generalization to the multivariate case. This generalization is the core of the dissertation and its main contribution to science. A well known representation of the asymmetric univariate Laplace distribution is used as the starting point. Properties of the new family of distributions are described in detail and parameters are estimated using the method of moments. In the final part of the dissertation an application of the new family of distributions to modelling financial data in the case of bivariate currency exchange rate data set is given. This new family shows the potential for modelling purposes.

Keywords: asymmetric Laplace law; double Weibull distribution; multivariate asymmetric Weibull distribution; currency exchange rate modelling

Povzetek

Disertacija obravnava nesimetrične Weibullove porazdelitve tako v eni kot v več dimenzijah. Predstavimo deloma nove definicije teh porazdelitev in izpeljemo metode za ocenjevanje parametrov, kar je nujna predpostavka za uporabo pri modeliranju finančnih podatkov kot so donosi finančnih naložb ali modeliranje menjalnih tečajev. Po obširnem pregledu v eni dimenziji je predstavljena posplošitev na več dimenzij. Ta posplošitev je dejanski prispevek disertacije. Posplošitev je posredna prek reprezentacij nesimetrične Laplaceove porazdelitve v eni dimenziji. Disertacija navaja lastnosti te nove družine porazdelitev in se loti tudi vprašanj ocenjevanja parametrov in simulacij. Disertacija se zaključi s finančno uporabo te nove družine porazdelitev na dejanskih menjalnih tečajih.

Ključne besede: asimetrična Laplaceova transformacija; dvojna Weibullova porazdelitev; večrazsežna asimetrična Weibullova porazdelitev; finančno modeliranje

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Introduction

The exponential distribution, whose standard probability density function (p.d.f.) is given by

$$f_E(x) = e^{-x}, \;\; x > 0, \tag{1}$$

is one of the most widely used distributions in science. Taking the $1/\alpha$ power of a standard exponential variable leads to classical Weibull distribution with the p.d.f.

$$f(x) = \alpha x^{\alpha-1} e^{-x^{\alpha}}, \;\; x > 0. \tag{2}$$

This distribution is used to model the breaking strength of materials (see Weibull [96, 97]) as well as in other applications, including quality control and reliability (see Weibull [98]). It is widely used probability distribution in finance, science and engineering (see Halinan [28], and Johnson at al. [40]). The distribution can be extended to the whole real line by symmetrization of the density (2), leading to the p.d.f.

$$f(x) = \frac{\alpha}{2}|x|^{\alpha-1} e^{-|x|^{\alpha}}, \;\; x \neq 0, \tag{3}$$

(see Balakrishnan and Kocherlakota [7]). This symmetric distribution is useful in modeling financial asset returns and in non-life insurance (see Chenyao et al. [9], Hürlimann [34], Mittnik and Rachev [74]).
The same distribution can be obtained by rising a Laplace variable with density

$$f(x) = \frac{1}{2} e^{-|x|}, \;\; x \in \mathbb{R}, \tag{4}$$

to the (signed) $1/\alpha$ power. Fernandez and Steel (see [19]) introduced skewness into the symmetric double Weibull distribution including two inverse scale factors (one on the positive and one on the negative half-axis) transforming a symmetric distribution with p.d.f. f into a skew one with the p.d.f.

$$g(x) = \frac{2\kappa}{1+\kappa^2} \begin{cases} f(x\kappa), & x \geq 0 \\ f(\frac{x}{\kappa}), & x < 0, \end{cases} \tag{5}$$

where $\kappa > 0$. Normal density generates the class of skew normal distributions (see Tiao and Lund [93], Mudholkar and Hutson [76], and references therein). The Laplace (double exponential) density (4) leads to the class of skew Laplace distributions, useful for stochastic modeling in variety of fields, including finance, economics, and the sciences (see Kotz et al. [49] and references therein). Ayebo and Kozubowski (see [6]) considered skew exponential power laws on $\mathbb{R}$, which generalize both the skew normal and the skew Laplace distributions.

Asymmetric double Weibull ($\mathcal{ADW}$) distributions are obtained following Fernandez and Steel approach (see [19]). One type of asymmetric distributions is obtained when the skewness is applied directly into the symmetric double Weibull distribution via (5). The other type arises from taking (signed) powers of skew Laplace laws. These two types of distributions have different forms and are discussed separately.
It is known that the standard Laplace random variable Y admits representation

$$Y \stackrel{d}{=} \sqrt{2E}Z, \tag{6}$$

where Z is a standard normal and E is a standard exponential random variable (see Kotz et al. [49]). This representation is a scale mixture of normal distributions (in other words, a normal distribution with a stochastic variance $2E$).
The question whether the double Weibull distribution itself is a scale mixture of normal distributions similar to (6) is considered. It turns out that answer is positive, but only in the case $0 < \alpha \leq 1$. For $\alpha = 1$, the double Weibull distribution reduces to the Laplace distribution, which is a scale mixture of normal distributions (see (6) above). It is known that for $\alpha < 1$, the classical Weibull variable $W = E^{\frac{1}{\alpha}}$ admits the representation

$$W \stackrel{d}{=} E/S, \tag{7}$$

where S is a *stable* random variable - a positive random variable with p.d.f. $f_S(s)$ and the Laplace transform

$$g(t) = \mathbb{E}e^{-tS} = \int_0^\infty e^{-st} f_S(s)ds = e^{-t^\alpha}, \tag{8}$$

(see Yannaros [95] and also Jurić [41]). Then the corresponding double Weibull variable X admits the representation

$$X \stackrel{d}{=} IW \stackrel{d}{=} IE/S \stackrel{d}{=} Y/S,$$

where Y is a standard Laplace variable and I is a variable taking on the values ± 1 with probabilities $1/2$. Thus, comparing with (6), we conclude that X is a mixture of normal distributions and the following new result summarizes this discussion,

$$X \stackrel{d}{=} \left(\frac{\sqrt{2E}}{S} \right) Z.$$

As already said, there are two ways of deriving an asymmetric double Weibull distribution. The first model, named "Asymmetric double Weibull distribution of type I" is obtained by introducing skewness into the symmetric (double) Weibull distribution using the approach of Fernandez and Steel (see [19]) and the fact that the classical Weibull variable can be represented as a power of an exponential variable.

Adding an additional scale parameter $\sigma > 0$, the following formulas for density and distribution function are obtained:

$$g(x) = \frac{1}{\sigma^\alpha}\frac{\alpha\kappa}{1+\kappa^2}\begin{cases}(\kappa x)^{\alpha-1}e^{-(\frac{x\kappa}{\sigma})^\alpha}, & x>0\\ (-\frac{x}{\kappa})^{\alpha-1}e^{-(-\frac{x}{\sigma\kappa})^\alpha}, & x<0\end{cases} \quad \text{and} \quad G(x) = \begin{cases}1-\frac{1}{1+\kappa^2}e^{-(\frac{x\kappa}{\sigma})^\alpha}, & x\geq 0\\ \frac{\kappa^2}{1+\kappa^2}e^{-(-\frac{x}{\sigma\kappa})^\alpha}, & x<0,\end{cases} \tag{9}$$

We denote the distribution of X by $\mathcal{ADW}_\alpha(\sigma,\kappa)$ and write $X \sim \mathcal{ADW}_\alpha(\sigma,\kappa)$.

To construct the second model, the operations of taking the power and introducing skewness are reversed. The steps for obtaining this model include symmetrization of the standard exponential random variable to obtain Laplace random variable, introducing the skewness into the Laplace p.d.f. via (5) to obtain a skew Laplace r.v. and taking (signed) $1/\alpha$ power. This model is named "Asymmetric double Weibull distribution of type II". The following formulas for density and corresponding distribution function are obtained:

$$g(x) = \frac{1}{\sigma}\frac{\alpha\kappa}{1+\kappa^2}\begin{cases}x^{\alpha-1}e^{-\frac{\kappa}{\sigma}x^\alpha}, & x>0\\ (-x)^{\alpha-1}e^{-\frac{1}{\kappa\sigma}(-x)^\alpha}, & x<0.\end{cases} \quad \text{and} \quad G_X(x) = \begin{cases}1-\frac{1}{1+\kappa^2}e^{-\frac{\kappa}{\sigma}x^\alpha}, & x\geq 0\\ \frac{\kappa^2}{1+\kappa^2}e^{-\frac{1}{\kappa\sigma}(-x)^\alpha}, & x<0.\end{cases} \tag{10}$$

We denote the distribution of X as $\mathcal{ADW}^*_\alpha(\sigma,\kappa)$ and write $X \sim \mathcal{ADW}^*_\alpha(\sigma,\kappa)$.
The most important contribution of this work is the extension of the univariate Weibull distribution to the multidimensional case. As already stated, the standard Laplace random variable Y admits the representation

$$Y \stackrel{d}{=} \sqrt{2E}Z,$$

where E is standard exponential and Z is standard normal variable independent of E. Furthermore, Kozubowski and Podgórski, (see [48]) show that the random variable

$$Y \stackrel{d}{=} mE + \sqrt{2E}Z \tag{11}$$

has the asymmetric Laplace distribution in the sense of (5), where m is a parameter of asymmetry. On the other hand, let L have the symmetric Laplace distribution and let S be an independent stable random variable with index $\alpha \in (0,1]$ defined by the Laplace transform (8). It can be shown by an elementary calculation that the random variable $Y = L/S$ has the symmetric Weibull distribution with parameters α and $\sigma = 1$. This together with (11) leads to the idea that the random variable W defined by

$$W \stackrel{d}{=} \frac{mE + \sqrt{2E}X}{S} \tag{12}$$

with independent E, $X \sim \mathrm{N}(0, \tau^2)$ and S, has the asymmetric double Weibull distribution. It can be verified by an elementary calculation that W indeed has the asymmetric Weibull distribution in the sense of (5) with parameters α, $\sigma = \tau$ and $\kappa = \frac{\sqrt{m^2+4\tau^2}-m}{2\tau}$.

The above representation leads to a multidimensional generalization of the univariate asymmetric Weibull distribution. We define

$$\mathbf{W} = \frac{\mathbf{m}E + \sqrt{2E}\mathbf{X}}{S}, \tag{13}$$

where $\mathbf{m} \in \mathbb{R}^d$, the quantity $\mathbf{\Sigma}$ is a $d \times d$ positive semi definite symmetric matrix, and the notation $\mathbf{X} \sim \mathrm{N}_d(\mathbf{0}, \mathbf{\Sigma})$ is used to indicate a d-dimensional normal distribution with the mean vector $\mathbf{0}$ and the covariance matrix $\mathbf{\Sigma}$. The marginal distributions of $\mathbf{W}$ are asymmetric Weibull as well. This justifies the name asymmetric multivariate Weibull distribution.

Symmetric and asymmetric, univariate and multivariate versions of classical (standard) Weibull and Laplace distributions have been used to model asset returns and currency exchange rates, (see [43], [52], [66], [74] and [83]). The above generalization provides a new family of distributions which can potentially be used in modeling multivariate financial data. The components will never be independent but the advantage of this generalization is in the fact that this family of distributions inherits nice properties of the multivariate normal distribution. Linear combinations of components are asymmetric Weibull. The other limitation is that $\alpha \in (0, 1]$ but it can be shown that only for such α we get a unimodal distribution which is of advantage for modeling purposes. Thus, the motivation for our work is to generalize asymmetric double univariate Weibull model and place it in the multidimensional setting. The model itself has an elegant form with nice properties showing a good application potential.

The thesis is organized as follows. First, in Chapter 1, we introduce skewness in symmetric (double) Weibull distribution following the approach of Fernandez and Steel (see [19]), and derive basic formulas for densities and cumulative distribution functions. Expressions for moments and other common parameters are provided as well.

In Chapter 2, we discuss estimation of the parameters and provide algebraic expressions for some of them. We also present asymptotic properties of the estimators for certain cases. For those parameters that could not be expressed analytically, we develop computer routines which are given in Appendices.

Chapter 3 is dedicated to practical applications in the univariate setting. Here, the asymmetric double Weibull distribution of type II along with three other distributions (Normal, Asymmetric Laplace and Exponential power) is applied to model currency exchange data set. The Kolmogorov-Smirnov distance is calculated to evaluate goodness of fit for each distribution. The results prove the Weibull model to be adequate.

In Chapter 4 a review of univariate and multivariate Weibull distributions based on

literature search is presented.
Chapter 5 discusses the representation of multivariate Weibull random vector defined by using the asymmetric Laplace and stable subordinator r.v. The connection of the multivariate model with type I univariate Weibull r.v. is shown. Properties obtained from the multivariate model such as the polar representation, linear transformation, conditional distribution, expected values and covariance matrices are presented as well. We continue with formula for density and a simulation algorithm accompanied with the graphical representation in Chapter 6.
Chapter 7 describes the quadrant probabilities in the bivariate case. In Chapter 8 estimation of the parameters based on method of moments is explained in detail. In order to obtain the parameters, numerical search using statistical package R is performed. The final part in Chapter 9 includes the application part of the bivariate case followed by a commentary and conclusions.

1 Asymmetric double Weibull distributions

This chapter starts with a review of the classical Weibull distribution and its fundamental properties. Next, an extension of the symmetric Weibull distribution to the whole real line, leading to the symmetric (double) Weibull distribution, is presented. We derive some basic properties along with the representation and the facts connected with stability and limiting properties. At the end, skewness is introduced into the symmetric distribution, leading to asymmetric double Weibull distributions of type I and II. The results of this chapter closely follow Jurić and Kozubowski [42].

1.1 Classical Weibull distribution

For $\alpha > 0$, the variable $Y = E^{\frac{1}{\alpha}}$,where E is a standard exponential random variable, has the standard Weibull distribution. With additional location and a scale parameter, we obtain a three-parameter Weibull distribution corresponding to the distribution of the random variable: $X = \sigma Y + \xi = \sigma E^{\frac{1}{\alpha}} + \xi$ denoted by $W_\alpha(\xi, \sigma)$. The three-parameter probability density function takes the form

$$f_X(x) = \frac{\alpha}{\sigma}\left(\frac{x-\xi}{\sigma}\right)^{\alpha-1} e^{-\left(\frac{x-\xi}{\sigma}\right)^\alpha}, x > \xi, \tag{14}$$

where $\alpha > 0$ is the shape parameter, $\sigma > 0$ is the scale parameter and $\xi \in \mathbb{R}$ is the location parameter. The corresponding cumulative distribution function is:

$$F_X(x) = 1 - e^{-\left(\frac{x-\xi}{\sigma}\right)^\alpha}, x > \xi. \tag{15}$$

The exponential distribution is obtained for $\alpha = 1$ and the Rayleigh distribution for $\alpha = 2$ as special cases. For $\alpha > 1$ there is a single mode at

$$x = \sigma\left(\frac{\alpha-1}{\alpha}\right)^{\frac{1}{\alpha}} + \xi. \tag{16}$$

This value tends to $\sigma + \xi$ as $\alpha \to \infty$. For $0 < \alpha \leq 1$ the mode is at ξ, and the density is a decreasing function of x for all $x > \xi$.

Since X^α has the standard exponential distribution, if X is Weibull with $\sigma = 1$ and $\xi = 0$, the n-th moment of X is the same as the $\frac{n}{\alpha}$-th moment of the standard exponential random variable,

$$E(X^n) = \Gamma\left(\frac{n}{\alpha} + 1\right). \tag{17}$$

The method of moments estimation gives the following equations for σ and α:

$$\hat{\sigma} = e^{a+\frac{\gamma}{\pi}\sqrt{6(b-a^2)}} \text{ and } \hat{\alpha} = \frac{\pi}{\sqrt{6(b-a^2)}},$$

where γ is the Euler constant, $a = \log\sigma - \frac{\gamma}{\alpha}$, and

$$b = (\log\sigma)^2 - 2\log\sigma(\log\sigma - a) + (\log\sigma - a)^2 + \left(\frac{\log\sigma - a}{\gamma}\right)^2 \frac{\pi^2}{6}.$$

The maximum likelihood estimator for σ leads to

$$\hat{\sigma} = \left(\frac{1}{n}\sum_{i=1}^{n} X_i^{\alpha}\right)^{\frac{1}{\alpha}} \tag{18}$$

when α is known. When this quantity is substituted into the log-likelihood function, this results in the function

$$h(\alpha) = n\left\{\log\alpha - \log\left(\frac{1}{n}\sum_{i=1}^{n}\log x_i^{\alpha}\right) - 1 + (\alpha-1)\frac{1}{n}\sum_{i=1}^{n}\log x_i\right\}, \tag{19}$$

which needs to be maximized (numerically) with respect to α. It can be shown that this procedures leads to unique estimates of both parameters. See Rockette at al. [85], Harter and Moore [32], [33], McCool [71], Pike [81] and Thoman at al. [92] for more information in this regard.

Many authors following Weibull (see Weibull [98], Kao [45],[46]) used this distribution in reliability and quality work. It is a flexible class of distributions whose hazard function can be decreasing, constant or increasing depending on the shape parameter α.

1.2 Balakrishnan and Kocherlakota double Weibull distribution

In this section we describe a symmetrization of Weibull distribution which was introduced by Balakrishnan and Kacherlakota (see [7]).
If a positive random variable X has density f, a random variable Y on $\mathbb{R} = (-\infty, \infty)$ can be defined with the density

$$h(x) = \frac{1}{2} f(|x|) \;\; x \in \mathbb{R}. \tag{20}$$

If f has the density of a standard exponential random variable (1), the symmetrization leads to a Laplace distribution with density (4) (see Kotz et al. [49]). The same procedure applied to the Weibull density (2) leads to a symmetric double Weibull distribution with density (3). Adding location and scale parameters, a three parameter double Weibull distribution introduced by Balakrishnan and Kocherlakota (see [7]) is obtained. Balakrishnan and Kocherlakota studied order statistics and linear estimation of parameters from this distribution (see also Dattareya Rao and Narasimhan [15]).
The density takes the form:

$$f(x) = \frac{\alpha}{2\sigma} \left| \frac{x-\xi}{\sigma} \right|^{\alpha-1} e^{-|\frac{x-\xi}{\sigma}|^{\alpha}}, x \in \mathbb{R}. \tag{21}$$

For $\xi = 0$, the above density which is denoted as $\mathcal{DW}_{\alpha}(\sigma)$, can be written as

$$f(x) = \frac{\alpha}{2\sigma} \begin{cases} (\frac{x}{\sigma})^{\alpha-1} e^{-(\frac{x}{\sigma})^{\alpha}}, & x \geq 0 \\ (-\frac{x}{\sigma})^{\alpha-1} e^{-(-\frac{x}{\sigma})^{\alpha}}, & x < 0, \end{cases} \tag{22}$$

while the corresponding distribution function is:

$$F(x) = \begin{cases} 1 - \frac{1}{2} e^{-(\frac{x}{\sigma})^{\alpha}}, & x \geq 0 \\ \frac{1}{2} e^{-(-\frac{x}{\sigma})^{\alpha}}, & x < 0 \end{cases} \tag{23}$$

(see Kotz et al. [49]).

It can be seen that the standard double Weibull distribution, $\mathcal{DW}_{\alpha}$, with density (3), is unimodal with the mode at 0 if $\alpha \leq 1$. When $\alpha > 1$, the distribution is bimodal with two modes located symmetrically on each side of the origin,

$$m_1 = \left(\frac{\alpha-1}{\alpha} \right)^{\frac{1}{\alpha}} \text{ and } m_2 = -\left(\frac{\alpha-1}{\alpha} \right)^{\frac{1}{\alpha}}. \tag{24}$$

Moments and related parameters are obtained from those of the classical Weibull distribution (see Jurić [41]). If $X \sim \mathcal{DW}_\alpha(1)$, then the n-th moment of X exists for $n > -\alpha$ and equals

$$\mathbb{E}(X^n) = \begin{cases} \Gamma\left(\frac{n}{\alpha}+1\right), & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \tag{25}$$

The absolute moment of order η is

$$\mathbb{E}(|X|^\eta) = \Gamma\left(\frac{\eta}{\alpha}+1\right), \ \eta > -\alpha. \tag{26}$$

In particular the mean is zero and the variance is $\Gamma(1+2/\alpha)$. The distribution and the quantile functions can be determined directly from the p.d.f. as:

$$F(x) = \begin{cases} 1-\frac{1}{2}e^{-x^\alpha}, & x \geq 0 \\ \frac{1}{2}e^{-(-x)^\alpha}, & x < 0. \end{cases} \quad \text{and} \quad F^{-1}(\rho) = \begin{cases} (-\log[2(1-\rho)])^{\frac{1}{\alpha}}, & \text{if } \rho \geq \frac{1}{2}, \\ -[-\log(2\rho)]^{\frac{1}{\alpha}}, & \text{if } \rho < \frac{1}{2}. \end{cases} \tag{27}$$

1.2.1 Representations

By construction, a standard double Weibull variable $X \sim \mathcal{DW}_\alpha(1)$ admits the following representations in terms of the standard classical Weibull and exponential variables (W and E, respectively):

$$X \stackrel{d}{=} IW \stackrel{d}{=} IE^{\frac{1}{\alpha}}. \tag{28}$$

Here, I is a variable taking on the values ± 1 with probabilities $1/2$, and all variables are mutually independent. Note that the absolute value of X has the classical Weibull distribution. On the right hand-side of (28), we can first symmetrize the exponential distribution, obtaining the Laplace variable $Y = IE$ with density (4), and subsequently take the $1/\alpha$ (signed) power of Y (see Jurić [41]). This leads to the representation

$$X \stackrel{d}{=} Y^{<\frac{1}{\alpha}>}, \tag{29}$$

where for $a \in \mathbb{R}$,

$$x^{<a>} = |x|^a \text{sign}(x) = \begin{cases} x^a, & x \geq 0 \\ -|x|^a, & x < 0 \end{cases}$$

is the signed power function.

Recall that the standard Laplace random variable Y admits the representation

$$Y \stackrel{d}{=} \sqrt{2E}Z, \tag{30}$$

where Z is a standard normal and E is a standard exponential random variable (see, Kotz et al. [49]). Thus, one more representation of $X \sim \mathcal{DW}_\alpha$ can be derived, this time related to the normal distribution:

$$X \stackrel{d}{=} (\sqrt{2E}Z)^{<\frac{1}{\alpha}>} = 2^{\frac{1}{2\alpha}} E^{\frac{1}{2\alpha}} Z^{<\frac{1}{\alpha}>}. \tag{31}$$

For $\alpha < 1$, the classical Weibull variable $W = E^{\frac{1}{\alpha}}$ admits the representation

$$W \stackrel{d}{=} E/S, \tag{32}$$

where S is a *stable* random variable - a positive random variable with p.d.f. $f_S(s)$ and the Laplace transform (8), (see Yannaros [95] and also Jurić and Kozubowski [42]). Then the double Weibull variable X can be represented as:

$$X \stackrel{d}{=} IW \stackrel{d}{=} IE/S \stackrel{d}{=} Y/S,$$

where Y is a standard Laplace variable. It can be seen that X is a mixture of normal distributions.

Finally, consider the case $\alpha > 1$, and suppose that $X \sim \mathcal{DW}_\alpha(1)$ is a scale mixture of normal distributions, that is $X \stackrel{d}{=} TZ$, where T and Z are independent, Z is standard normal, and T is some non-negative random variable. Note that the distribution of Z is unimodal with the mode at zero. Since this property holds if and only if the relevant variable can be represented as a product of two independent variables, of which one is standard uniform (see Shepp [88]), it follows that $Z \stackrel{d}{=} VU$, where V and U are independent and U is standard uniform. Thus, we also have $X \stackrel{d}{=} (TV)U$, showing in turn, that X is unimodal with the mode at zero as well. However, when $\alpha > 1$, X has two modes (24), so it can not have the above representation. Therefore, we have the following result.

Proposition 1.1 *Let $X \sim \mathcal{DW}_\alpha$. Then, the distribution of X is a scale mixture of normal distributions if and only if $0 < \alpha \leq 1$. Moreover, for $0 < \alpha < 1$ we have*

$$X \stackrel{d}{=} \left(\frac{\sqrt{2E}}{S}\right) Z,$$

where all variables on the right-hand side are independent, E is standard exponential, Z is standard normal, and S is a stable random variable with distribution function. (8).

Proof. We need to prove that $(\frac{\sqrt{2E}}{S})Z$ is symmetric Weibull. Define a function $\Phi : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^3$ as

$$\Phi(u, s, z) = (u, s, \frac{\sqrt{2u}}{s} z). \tag{33}$$

Then,

$$\Phi^{-1}(u, s, x) = (u, s, \frac{sx}{\sqrt{2u}}), \tag{34}$$

and Jacobian

$$J_{\Phi^{-1}}(u, s, x) = \frac{s}{\sqrt{2u}}. \tag{35}$$

A straightforward application of the transformation formula gives the p.d.f. of r.v. (E, S, X):

$$f_{E,S,X}(u, s, x) = e^{-u} f_S(s) \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2x^2}{4u}} \frac{s}{\sqrt{2u}}. \tag{36}$$

By integrating over u and taking into account Oberhettinger and Badii 5.28 formula

$$\int_0^\infty \frac{1}{\sqrt{t}} e^{-\frac{a}{t} - pt} dt = \left(\frac{\pi}{p}\right)^{\frac{1}{2}} e^{-2\sqrt{ap}} \tag{37}$$

(see [79]), we get

$$f_{S,X}(s, x) = \frac{s}{2} f_S(s) e^{-s|x|} \tag{38}$$

Take the derivative with respect to y in

$$\int_0^\infty e^{-ys} f_S(s) ds = e^{-y^\alpha} \tag{39}$$

to get

$$\int_0^\infty s e^{-ys} f_S(s) ds = \alpha y^{\alpha-1} e^{-y^\alpha} \tag{40}$$

we have

$$f_X(x) = \frac{\alpha}{2} |x|^{\alpha-1} e^{-|x|^\alpha}. \tag{41}$$

1.3 Stability properties

Some interesting properties of $\mathcal{DW}_\alpha$ variables are presented in this section. For $Y_1,\dots,Y_n$ being i.i.d.classical Laplace random variables, the following relation holds:

$$p^{\frac{1}{2}}(Y_1 + Y_2 + \cdots + Y_{N_p}) \stackrel{d}{=} Y_1,$$

where N_p is a geometric random variable with the probability mass function

$$\mathbb{P}(N_p = k) = (1-p)^{k-1}p, \;\; k = 1, 2, 3\dots$$

and is independent from the $Y_i's$ (see Kotz et al. [49]).
Since $X_i \stackrel{d}{=} Y_i^{<\frac{1}{\alpha}>}$ and $X_i^{<\alpha>} \stackrel{d}{=} Y_i$ where $X_1,\dots,X_n$ are i.i.d. random variables with the $\mathcal{DW}_\alpha$ distribution, the following relation can be obtained:

$$p^{\frac{1}{2}}(X_1^{<\alpha>} + X_2^{<\alpha>} + \cdots + X_{Np}^{<\alpha>}) \stackrel{d}{=} X_1^{<\alpha>}.$$

Also,

$$p^{\frac{1}{2}}X_1^{<\alpha>} + (1-I)X_2^{<\alpha>} \stackrel{d}{=} X_1^{<\alpha>}$$
$$p^{\frac{1}{2}}IX_1^{<\alpha>} + (1-I)(X_2^{<\alpha>} + p^{\frac{1}{2}}X_3^{<\alpha>}) \stackrel{d}{=} X_1^{<\alpha>},$$

(see Jurić and Kozubowski [42]), where X_1, X_2, and X_3 are i.i.d. $\mathcal{DW}_\alpha$ variables, I has a Bernoulli distribution with $\mathbb{P}(I = 1) = 1 - \mathbb{P}(I = 0) = p$, and all variables in these relations are mutually independent.
Finally, the property of stability with respect to the operation of minimum is derived,

$$n^{\frac{1}{\alpha}} \min(|X_1|, |X_2|, \dots |X_n|) \stackrel{d}{=} |X_1|,$$

which follows from the same property of the Weibull distribution. The importance of the above stability properties is discussed in the context of modeling financial data (see Mittnik and Rachev [74]).

1.4 Asymmetric double Weibull distribution of type I

In this part skewness is introduced into the symmetric double Weibull distribution following the approachof Fernandez and Steel, (see [19]), showing two different ways of obtaining asymmetric double Weibull distribution. In Chapter 5, the multivariate generalization of this asymmetric Weibull distribution will be considered showing that the marginal distribution of the multivariate Weibull distribution is Type I. The asymmetric double Weibull distribution of type I arises from the following steps:

- Start with a standard exponential random variable E with density (1).
- Convert E into a standard classical Weibull random variable $W = E^{\frac{1}{\alpha}}$ with the p.d.f (2).
- Convert the density of W into a double Weibull density function f given by (3).
- Apply the procedure (5) of Fernandez and Steel (see [19]) to the f above.

With an additional scale parameter $\sigma > 0$, the following definition of an asymmetric double Weibull distribution of type I is obtained:

Definition 1.1 *A random variable X is said to have an asymmetric double Weibull distribution of type I ($\mathcal{ADW}$) if for parameters $\alpha > 0$, $\sigma > 0$ and $\kappa > 0$ the density function of X has the form*

$$g(x) = \frac{1}{\sigma^\alpha}\frac{\alpha\kappa}{1+\kappa^2}\begin{cases} (\kappa x)^{\alpha-1}e^{-(\frac{x\kappa}{\sigma})^\alpha}, & x > 0 \\ (-\frac{x}{\kappa})^{\alpha-1}e^{-(-\frac{x}{\sigma\kappa})^\alpha}, & x < 0 \end{cases} \tag{42}$$

We denote the distribution of X by $\mathcal{ADW}_\alpha(\sigma, \kappa)$ and write $X \sim \mathcal{ADW}_\alpha(\sigma, \kappa)$.

The distribution function can be easily computed as follows:

$$G(x) = \begin{cases} 1 - \frac{1}{1+\kappa^2}e^{-(\frac{x\kappa}{\sigma})^\alpha}, & x \geq 0 \\ \frac{k^2}{1+\kappa^2}e^{-(-\frac{x}{\sigma\kappa})^\alpha}, & x < 0. \end{cases} \tag{43}$$

If the following notation is used,

$$x^+ = \begin{cases} x, & x > 0 \\ 0, & x \leq 0 \end{cases} \quad \text{and } x^- = \begin{cases} 0, & x > 0 \\ -x, & x < 0, \end{cases} \tag{44}$$

the p.d.f. of corresponding $\mathcal{ADW}_\alpha(\sigma, \kappa)$ distribution can be written in a more compact way:

$$g(x) = \frac{1}{\sigma^\alpha}\frac{\alpha\kappa}{1+\kappa^2}\left[(x^+\kappa)^{\alpha-1}e^{-(\frac{\kappa}{\sigma}x^+)^\alpha} + \left(\frac{x^-}{\kappa}\right)^{\alpha-1}e^{-(\frac{x^-}{\kappa\sigma})^\alpha}\right], \quad x \neq 0.$$

Some limiting cases can be emphasized:

- For $\sigma = 0$ and $\kappa > 0$, the distribution is degenerate at zero, by taking the limit of the c.d.f. as $\sigma \to 0^+$.

- For $\kappa = 0$ and $\sigma > 0$ we do not have a proper distribution, since the limit of the c.d.f. as $\kappa \to 0^+$ (with σ held fixed) is equal to 0 for every $x \neq 0$. However, when $\kappa \to 0^+$ and $\sigma = a\kappa \to 0^+$, where $a > 0$ is some constant, then the $\mathcal{ADW}_\alpha(\sigma, \kappa)$ distribution converges weakly to the classical Weibull distribution with shape parameter a.

- If $\kappa \to \infty$ and σ is fixed, then the c.d.f. approaches 1 at each x that is not zero, which again is not a valid c.d.f. However, when $\kappa \to \infty$ and $\sigma = \frac{a}{\kappa} \to 0^+$, where $a > 0$ is some constant, then the $\mathcal{ADW}_\alpha(\sigma, \kappa)$ distribution converges weakly to the distribution of $-W$, where W has the classical Weibull distribution with shape parameter a.

In Figure 1 graphs of selected $\mathcal{ADW}_\alpha$ densities with $\kappa = 1.5$, $\sigma = 1$, $\alpha = 0.9$, 1, 1.5, and 3 respectively are presented.The figure is reprinted from [42].

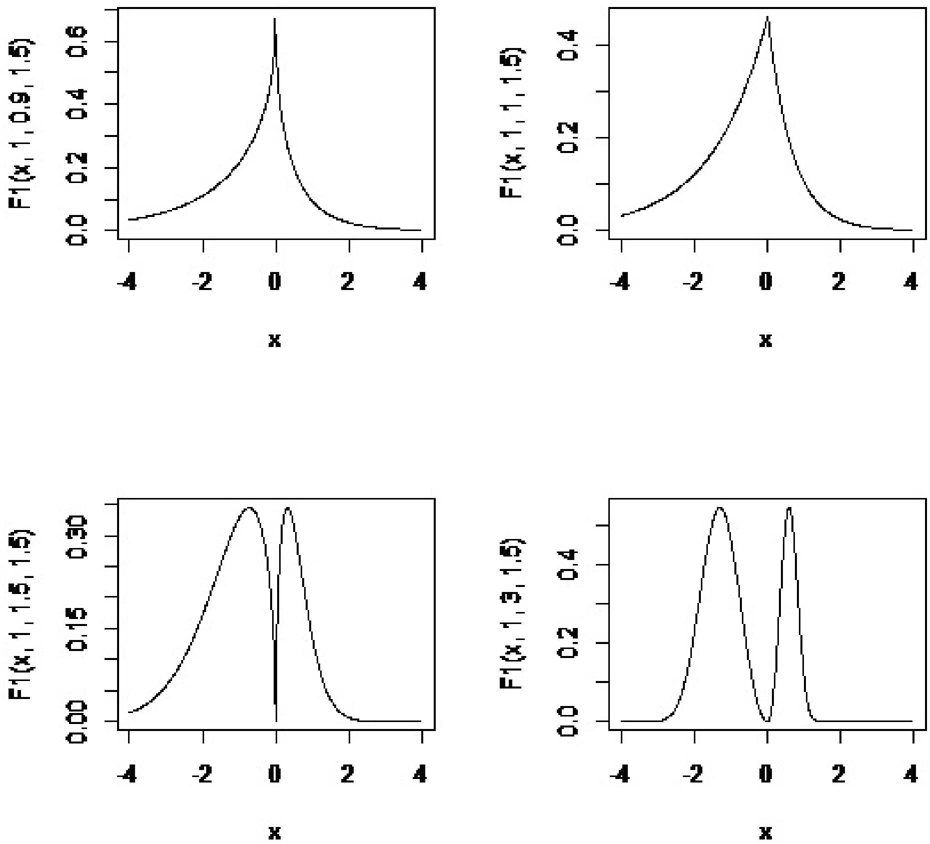

Figure 1: Graphs of asymmetric double Weibull densities $\mathcal{ADW}_\alpha$

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1.5 Asymmetric double Weibull distribution of type II

The distribution is constructed following the steps:

- Start with a standard exponential random variable E .
- Symmetrize the density of E to obtain a double exponential (Laplace) density function (4).
- Introduce a skewness into the Laplace p.d.f. using (5) to obtain a skew Laplace r.v. Y with the p.d.f.

$$f(x) = \frac{1}{\sigma}\frac{\kappa}{1+\kappa^2}\begin{cases} e^{\frac{-x\kappa}{\sigma}}, & x > 0 \\ e^{\frac{x}{\kappa\sigma}}, & x < 0, \end{cases} \tag{45}$$

  where we also included an additional scale parameter $\sigma > 0$ (see Kozubowski and Podgórski [51]).

- Take the (signed) $1/\alpha$ power of Y.

This leads to the following result.

Definition 1.2 *A random variable X is said to have an asymmetric double Weibull distribution of type II $\mathcal{ADW}^*$ if for parameters $\alpha > 0$, $\sigma > 0$ and $\kappa > 0$ the denity function of X has the form*

$$g(x) = \frac{1}{\sigma}\frac{\alpha\kappa}{1+\kappa^2}\begin{cases} x^{\alpha-1}e^{-\frac{\kappa}{\sigma}x^\alpha}, & x > 0 \\ (-x)^{\alpha-1}e^{-\frac{1}{\kappa\sigma}(-x)^\alpha}, & x < 0. \end{cases} \tag{46}$$

We denote the distribution of X by $\mathcal{ADW}^*_\alpha(\sigma,\kappa)$ and write $X \sim \mathcal{ADW}^*_\alpha(\sigma,\kappa)$.

The corresponding distribution function is

$$G_X(x) = \begin{cases} 1 - \frac{1}{1+\kappa^2}e^{-\frac{\kappa}{\sigma}x^\alpha}, & x \geq 0 \\ \frac{\kappa^2}{1+\kappa^2}e^{-\frac{1}{\kappa\sigma}(-x)^\alpha}, & x < 0. \end{cases} \tag{47}$$

Using the notation x^+ and x^- described previously in (44), the p.d.f. of the $\mathcal{ADW}^*$ distribution can be written in a more compact way:

$$g(x) = \frac{1}{\sigma}\frac{\alpha\kappa}{1+\kappa^2}\left[(x^+)^{\alpha-1}e^{-\frac{\kappa}{\sigma}(x^+)^\alpha} + (x^-)^{\alpha-1}e^{-\frac{1}{\sigma\kappa}(x^-)^\alpha}\right], \ \ x \neq 0.$$

As in the case of the $\mathcal{ADW}$ distribution, the following limit cases can be emphasized.

- When $\sigma = 0$ and $\kappa > 0$ we obtain degenerate distribution at zero (corresponding to the random variable $X = 0$).

- If $\kappa \to 0^+$ and $\sigma = a\kappa \to 0$ for some constant $a > 0$, the $\mathcal{ADW}^*(\sigma, \kappa)$ distribution converges weakly to the classical Weibull distribution with scale parameter $a^{1/\alpha}$.

- If $\kappa \to \infty$ and $\sigma = \frac{a}{\kappa} \to 0^+$ for some $a > 0$, the $\mathcal{ADW}^*(\sigma, \kappa)$ distribution converges weakly to the distribution of $-W$, where W has the classical Weibull distribution with scale parameter $a^{1/\alpha}$.

Figure 2 contains selected $\mathcal{ADW}_\alpha{}^*$ densities with $\kappa = 1.5$, $\sigma = 1$, $\alpha = 0.9$, 1, 1.5, and 3 respectively.The figure is reprinted from [42].

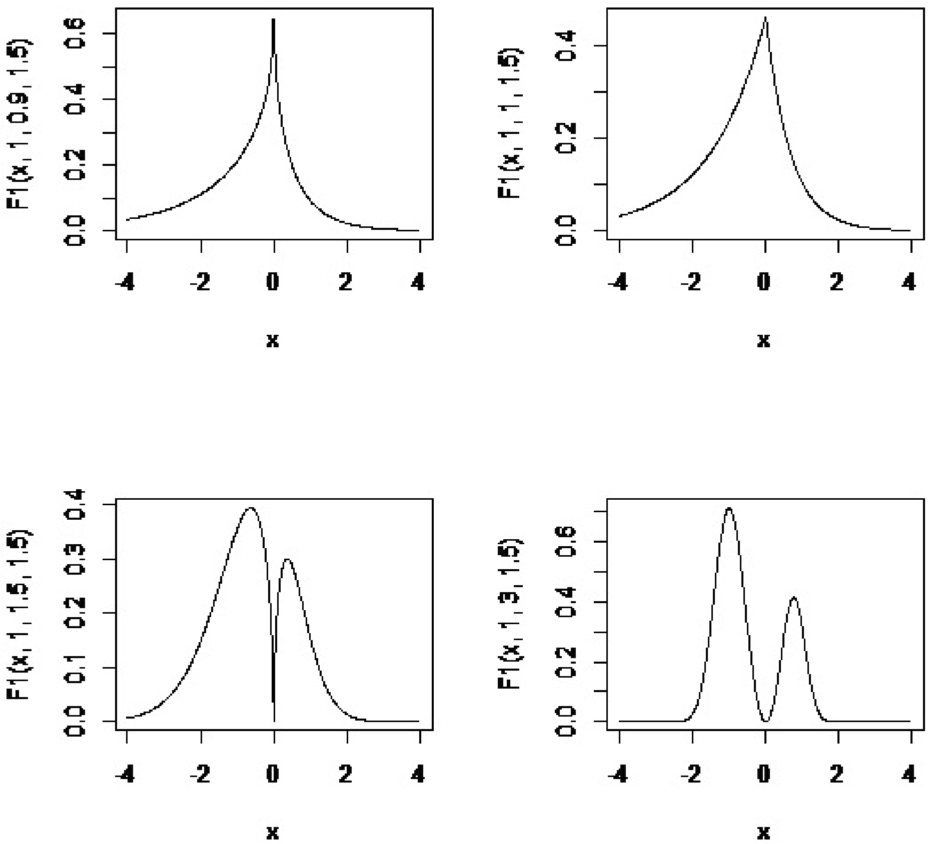

Figure 2: Graphs of asymmetric double Weibull densities $\mathcal{ADW}_\alpha^*$

1.6 Connection between the two asymmetric double Weibull distributions

If $X \sim \mathcal{ADW}_\alpha(1,\kappa)$ and $Y \sim \mathcal{ADW}^*_\alpha(1,\kappa)$, assuming α's being equal and denoting c.d.f.'s by $G(x)$ and $G^*(x)$ respectively, the probability integral transformation (see DeGroot [16], p. 154) yields the following result:

$$Y \stackrel{d}{=} (G^*)^{-1}(G(X)).$$

The quantile function of $Y \sim \mathcal{ADW}^*(1,\kappa)$ is of the form (see Jurić [41]):

$$(G^*)^{-1}(\rho) = \begin{cases} -[-\kappa \log(\rho \frac{1+(\kappa)^2}{(\kappa)^2})]^{\frac{1}{\alpha}}, & 0 < \rho < \frac{(\kappa)^2}{1+(\kappa)^2} \\ \left\{-\frac{1}{\kappa}\log[(1+(\kappa)^2)(1-\rho)]\right\}^{\frac{1}{\alpha}}, & \frac{(\kappa)^2}{1+(\kappa)^2} \le \rho < 1, \end{cases} \tag{48}$$

which combining with G given by (43) gives

$$Y \stackrel{d}{=} \kappa^{(1-\frac{1}{\alpha})\mathrm{sign}(X)} X.$$

If $X \sim \mathcal{ADW}_\alpha(\sigma,\kappa)$ and $Y \sim \mathcal{ADW}^*_\alpha(\sigma,\kappa)$, the following connection is obtained:

$$Y \stackrel{d}{=} \sigma^{\frac{1}{\alpha}-1} \kappa^{(1-\frac{1}{\alpha})\mathrm{sign}(X)} X. \tag{49}$$

Note that for $\alpha = 1$, $Y \stackrel{d}{=} X$. It can be concluded that the two ways of getting a skew double Weibull distribution lead to the same distribution.

1.7 Moments and related parameters

In this section moments and related parameters of asymmetric double Weibull distributions are presented. The proofs can be found in Jurić (see [41]).

1.7.1 Asymmetric double Weibull distribution of type I

If $X \sim \mathcal{ADW}_\alpha(\sigma,\kappa)$, then the following results for the moments are obtained:

$$\mathbb{E}X^n = \frac{\kappa^2}{1+\kappa^2}(-\sigma\kappa)^n \Gamma\left(\frac{n}{\alpha}+1\right) + \frac{1}{1+\kappa^2}\left(\frac{\sigma}{\kappa}\right)^n \Gamma\left(\frac{n}{\alpha}+1\right), \quad -\alpha < n \in \mathbb{Z} \tag{50}$$

$$\mathbb{E}|X|^\eta = \frac{\kappa^{\eta+2}}{1+\kappa^2}\sigma^\eta \Gamma\left(\frac{\eta}{\alpha}+1\right) + \frac{1}{1+\kappa^2}\left(\frac{\sigma}{\kappa}\right)^\eta \Gamma\left(\frac{\eta}{\alpha}+1\right), \quad -\alpha < \eta \in \mathbb{R} \tag{51}$$

$$\mathbb{E}(X^+)^\eta = \left(\frac{\sigma}{\kappa}\right)^\eta \frac{1}{1+\kappa^2}\Gamma\left(\frac{\eta}{\alpha}+1\right), \quad -\alpha < \eta \in \mathbb{R} \tag{52}$$

$$\mathbb{E}(X^-)^\eta = (\sigma\kappa)^\eta \frac{\kappa^2}{1+\kappa^2}\Gamma\left(\frac{\eta}{\alpha}+1\right), \quad -\alpha < \eta \in \mathbb{R} \tag{53}$$

In particular,

$$\mathbb{E}X = \frac{\sigma}{\kappa}\frac{1-\kappa^4}{1+\kappa^2}\Gamma\left(\frac{1}{\alpha}+1\right) = \frac{\sigma}{\kappa}(1-\kappa^2)\Gamma\left(\frac{1}{\alpha}+1\right), \tag{54}$$

$$\mathbb{V}\mathrm{ar}(X) = \frac{\sigma^2}{\kappa^2}\left\{\frac{1+\kappa^6}{1+\kappa^2}\Gamma\left(\frac{2}{\alpha}+1\right) - (1-\kappa^2)^2\Gamma^2\left(\frac{1}{\alpha}+1\right)\right\}, \tag{55}$$

and the coefficient of skewness is

$$\gamma_1 = (1-\kappa^2)\frac{(1+\kappa^4)\Gamma(\frac{3}{\alpha}+1) - 3(1-\kappa^2+\kappa^4)\Gamma(\frac{1}{\alpha}+1)\Gamma(\frac{2}{\alpha}+1) + 2(1-\kappa^2)^2\Gamma^3(\frac{1}{\alpha}+1)}{\left\{(1-\kappa^2+\kappa^4)\Gamma(\frac{2}{\alpha}+1) - (1-\kappa^2)^2\Gamma^2(\frac{1}{\alpha}+1)\right\}^{\frac{3}{2}}}.$$

The quantiles can be derived as:

$$x_\rho = \begin{cases} \frac{\sigma}{\kappa}\left[\log\frac{1}{(1+\kappa^2)(1-\rho)}\right]^{\frac{1}{\alpha}}, & \rho \geq \frac{\kappa^2}{1+\kappa^2} \\ -\sigma\kappa\left[\log\frac{\kappa^2}{\rho(1+\kappa^2)}\right]^{\frac{1}{\alpha}}, & \rho < \frac{\kappa^2}{1+\kappa^2}. \end{cases} \tag{56}$$

Median is easily computed from the above equation:

$$M = \begin{cases} \frac{\sigma}{\kappa}\left[\log\frac{2}{1+\kappa^2}\right]^{\frac{1}{\alpha}}, & \kappa \in (0,1] \\ -\sigma\kappa\left[\log\frac{2\kappa^2}{1+\kappa^2}\right]^{\frac{1}{\alpha}}, & \kappa \in (1,\infty). \end{cases} \tag{57}$$

Discussion about modes is summarized in the following Proposition.

Proposition 1.2 *If $X \sim \mathcal{ADW}_\alpha(\sigma,\kappa)$ with density (9) then,*
(i) For $\alpha \leq 1$ the distribution is unimodal with the mode at 0. Moreover, the value of the density at the mode is

$$g(0) = \begin{cases} \infty, & \alpha < 1 \\ \frac{\kappa}{\sigma(1+\kappa^2)}, & \alpha = 1. \end{cases}$$

(ii) For $\alpha > 1$ the distribution is bimodal with the two modes,

$$m_1 = -\sigma\kappa\left(\frac{\alpha-1}{\alpha}\right)^{\frac{1}{\alpha}} < 0 \text{ and } m_2 = \frac{\sigma}{\kappa}\left(\frac{\alpha-1}{\alpha}\right)^{\frac{1}{\alpha}} > 0.$$

Moreover, we have $g(0) = 0$ and

$$g(m_1) = g(m_2) = \frac{\alpha}{\sigma}\frac{\kappa}{1+\kappa^2}\left(\frac{\alpha-1}{\alpha}\right)^{\frac{\alpha-1}{\alpha}} e^{-(\frac{\alpha-1}{\alpha})}.$$

Our next result provides the entropy of asymmetric double Weibull distribution.

Proposition 1.3 *Let X have an $\mathcal{ADW}_\alpha(\sigma,\kappa)$ distribution with density f given by (9). Then the entropy of X is*

$$H(X) = \mathbb{E}\{-\log f(X)\} = -\log\left(\frac{1}{\sigma}\frac{\alpha\kappa}{1+\kappa^2}\right) + \frac{\gamma(\alpha-1)}{\alpha} + 1, \tag{58}$$

where $\gamma \approx 0.5772$ is the Euler constant.

1.7.2 Asymmetric double Weibull distribution of type II

For $X \sim \mathcal{ADW}^*_\alpha(\sigma,\kappa)$, th following results for the moments are obtained:

$$\mathbb{E}X^n = \frac{\kappa^2}{1+\kappa^2}(-1)^n(\sigma\kappa)^{\frac{n}{\alpha}}\Gamma\left(\frac{n}{\alpha}+1\right) + \frac{1}{1+\kappa^2}\left(\frac{\sigma}{\kappa}\right)^{\frac{n}{\alpha}}\Gamma\left(\frac{n}{\alpha}+1\right), \quad -\alpha < n \in \mathbb{Z} \tag{59}$$

$$\mathbb{E}|X|^\eta = \frac{\kappa^2}{1+\kappa^2}(\sigma\kappa)^{\frac{\eta}{\alpha}}\Gamma\left(\frac{\eta}{\alpha}+1\right) + \frac{1}{1+\kappa^2}(\frac{\sigma}{\kappa})^{\frac{\eta}{\alpha}}\Gamma\left(\frac{\eta}{\alpha}+1\right), \quad -\alpha < \eta \in \mathbb{R} \tag{60}$$

$$\mathbb{E}(X^+)^\eta = \frac{1}{1+\kappa^2}\left(\frac{\sigma}{\kappa}\right)^{\frac{\eta}{\alpha}}\Gamma\left(\frac{\eta}{\alpha}+1\right), \quad \eta > -\alpha \tag{61}$$

$$\mathbb{E}(X^-)^\eta = \frac{\kappa^2}{1+\kappa^2}(\kappa\sigma)^{\frac{\eta}{\alpha}}\Gamma\left(\frac{\eta}{\alpha}+1\right), \quad \eta > -\alpha. \tag{62}$$

In particular,

$$\mathbb{E}X = \frac{\Gamma(\frac{1}{\alpha}+1)}{1+\kappa^2}\left(\frac{\sigma}{\kappa}\right)^{\frac{1}{\alpha}}(1-\kappa^{2+\frac{2}{\alpha}}), \tag{63}$$

$$\mathbb{V}\text{ar}X = \frac{1}{1+\kappa^2}\left(\frac{\sigma}{\kappa}\right)^{\frac{2}{\alpha}}\left\{(1+\kappa^{2+\frac{4}{\alpha}})\Gamma\left(\frac{2}{\alpha}+1\right) - \frac{(1-\kappa^{2+\frac{2}{\alpha}})^2}{1+\kappa^2}\Gamma^2\left(\frac{1}{\alpha}+1\right)\right\}, \tag{64}$$

and the coefficient of skewness is

$$\gamma_1 = \frac{(1+\kappa^2)^2(1-\kappa^{2+\frac{6}{\alpha}})\Gamma(\frac{3}{\alpha}+1) - 3(1+\kappa^2)(1-\kappa^{2+\frac{2}{\alpha}})(1+\kappa^{2+\frac{4}{\alpha}})\Gamma(\frac{1}{\alpha}+1)\Gamma(\frac{2}{\alpha}+1)}{\left\{(1+\kappa^2)(1+\kappa^{2+\frac{4}{\alpha}})\Gamma(\frac{2}{\alpha}+1) - (1-\kappa^{2+\frac{2}{\alpha}})^2\Gamma^2(\frac{1}{\alpha}+1)\right\}^{\frac{3}{2}}} +$$

$$+\frac{2(1-\kappa^{2+\frac{2}{\alpha}})^3\Gamma^3(\frac{1}{\alpha}+1)}{\left\{(1+\kappa^2)(1+\kappa^{2+\frac{4}{\alpha}})\Gamma(\frac{2}{\alpha}+1) - (1-\kappa^{2+\frac{2}{\alpha}})^2\Gamma^2(\frac{1}{\alpha}+1)\right\}^{\frac{3}{2}}}.$$

The ρ-th quantile of the $\mathcal{ADW}^*_\alpha(\sigma,\kappa)$ distribution is

$$x_\rho = \begin{cases} \left[\frac{\sigma}{\kappa}\log\frac{1}{(1+\kappa^2)(1-\rho)}\right]^{\frac{1}{\alpha}}, & \rho \geq \frac{\kappa^2}{1+\kappa^2} \\ -[\sigma\kappa\log\frac{\kappa^2}{\rho(1+\kappa^2)}]^{\frac{1}{\alpha}}, & \rho < \frac{\kappa^2}{1+\kappa^2}. \end{cases} \tag{65}$$

In particular, the median is

$$M = \begin{cases} \left[\frac{\sigma}{\kappa}\log\frac{2}{1+\kappa^2}\right]^{\frac{1}{\alpha}}, & \kappa \in (0,1] \\ -\left[\sigma\kappa\log\frac{2\kappa^2}{1+\kappa^2}\right]^{\frac{1}{\alpha}}, & \kappa \in (1,\infty). \end{cases} \tag{66}$$

The following result concerns the modality.

Proposition 1.4 *If $X \sim \mathcal{ADW}^*_\alpha(\sigma,\kappa)$ with density (10) then*

(i) For $\alpha \le 1$ the distribution is unimodal with the mode at 0.The value of the density at the mode is

$$g(0) = \begin{cases} \infty, & \alpha < 1 \\ \frac{\kappa}{\sigma(1+\kappa^2)}, & \alpha = 1. \end{cases}$$

(ii) For $\alpha > 1$ the distribution is bimodal with the two modes

$$m_1 = -\left(\sigma\kappa\frac{\alpha-1}{\alpha}\right)^{\frac{1}{\alpha}} < 0 \ \text{ and } \ m_2 = \left(\frac{\sigma}{\kappa}\frac{\alpha-1}{\alpha}\right)^{\frac{1}{\alpha}} > 0.$$

Moreover, we have $g(0) = 0$ and

$$g(m_1) = \frac{1}{\sigma}\frac{\alpha\kappa}{1+\kappa^2}(\sigma\kappa)^{\left(\frac{\alpha-1}{\alpha}\right)}\left(\frac{\alpha-1}{\alpha}\right)^{\left(\frac{\alpha-1}{\alpha}\right)} e^{-\left(\frac{\alpha-1}{\alpha}\right)},$$

$$g(m_2) = \frac{1}{\sigma}\frac{\alpha\kappa}{1+\kappa^2}\left(\frac{\sigma}{\kappa}\right)^{\left(\frac{\alpha-1}{\alpha}\right)}\left(\frac{\alpha-1}{\alpha}\right)^{\left(\frac{\alpha-1}{\alpha}\right)} e^{-\left(\frac{\alpha-1}{\alpha}\right)}.$$

It is interesting, that unlike the $\mathcal{ADW}$ case, here the values at the modes in case $\alpha > 1$ are not the same (unless $\kappa = 1$ and distribution is symmetric).

Finally, we have

Proposition 1.5 *Let X have an $\mathcal{ADW}^*_\alpha(\sigma,\kappa)$ distribution with density f given by (10). Then the entropy of X is*

$$H(X) = -\log\left(\frac{1}{\sigma}\frac{\alpha\kappa}{1+\kappa^2}\right) + \frac{\gamma(\alpha-1)}{\alpha} + 1 - \frac{\alpha-1}{\alpha}\left(\log\sigma + \frac{\kappa^2-1}{\kappa^2+1}\log\kappa\right), \tag{67}$$

where γ is the Euler constant.

We can use the connection between the two Weibull distributions given in (49) to obtain many properties of $Y \sim \mathcal{ADW}^*_\alpha(\sigma,\kappa)$ from those of $X \sim \mathcal{ADW}_\alpha(\sigma,\kappa)$.

1.8 Representations and simulation

An extension of the representation (28) to the skew case is straightforward (see Jurić [41]). A standard conditioning argument shows that in case $X \sim \mathcal{ADW}_\alpha(\sigma, \kappa)$ we have

$$X \stackrel{d}{=} \sigma I E^{\frac{1}{\alpha}}, \tag{68}$$

where E has a standard exponential distribution and

$$I = \begin{cases} -\kappa, & \text{with prob. } \frac{\kappa^2}{1+\kappa^2} \\ \frac{1}{\kappa}, & \text{with prob. } \frac{1}{1+\kappa^2}. \end{cases} \tag{69}$$

The following algorithm for generating random variates from this distribution is presented.

An $\mathcal{ADW}_\alpha(\sigma, \kappa)$ **generator**.

- Generate a standard exponential variable E
- Generate a standard uniform $[0, 1]$ random variable U
- If $U < \frac{\kappa^2}{\kappa^2+1}$, set $I \leftarrow -\kappa$ else set $I \leftarrow \kappa^{-1}$
- Set $X \leftarrow \sigma I E^{\frac{1}{\alpha}}$
- Return X

A similar representation of a skew Weibull variable of type II,

$$X \stackrel{d}{=} \sigma^{1/\alpha} I^{<1/\alpha>} E^{\frac{1}{\alpha}}, \tag{70}$$

where I and E are as before, produced the following random variate generator based on the above representation.

An $\mathcal{ADW}^*_\alpha(\sigma, \kappa)$ **generator**.

- Generate a standard exponential variate E
- Generate a standard uniform $[0, 1]$ random variate U
- If $U < \frac{\kappa^2}{\kappa^2+1}$ set $J \leftarrow -\kappa^{\frac{1}{\alpha}}$, else set $J \leftarrow \kappa^{-\frac{1}{\alpha}}$
- Set $X \leftarrow \sigma^{\frac{1}{\alpha}} J E^{\frac{1}{\alpha}}$
- Return X

Remark. The above representations are useful in showing that the distribution of $|X|$, where X has a skew Weibull distribution, is a mixture of two classical Weibull distributions. For example, if $X \sim \mathcal{ADW}^*_\alpha(1, \kappa)$, then

$$|X| = \kappa^{1/\alpha} I W_1 + \kappa^{-1/\alpha}(1 - I) W_2, \tag{71}$$

where I, W_1 and W_2 are independent, I takes on the values 1 and 0 with probabilities $\frac{\kappa^2}{1+\kappa^2}$ and $\frac{1}{1+\kappa^2}$, respectively, and W_1 and W_2 are two i.i.d. standard classical Weibull variables.

2 Estimation of the parameters - the univariate case

In this chapter we discuss maximum likelihood estimation of the skew double Weibull parameters. We focus on estimating σ and κ, assuming that the shape parameter α is known. We establish the existence and uniqueness of the estimators, and derive their asymptotic properties. We also briefly discuss a numerical procedure for estimating the parameter α. The results of this chapter closely follow Jurić and Kozubowski [43].

2.1 Asymmetric double Weibull distribution of Type I

For an i.i.d. random sample from an $\mathcal{ADW}_\alpha(\sigma, \kappa)$ distribution with density (9), the log-likelihood function takes the form:

$$\log L(\alpha, \sigma, \kappa) = n \Big\{ \log \alpha + (1 + (\alpha - 1)\bar{x}_{\text{sign}}) \log \kappa - \alpha \log \sigma - \log(1 + \kappa^2) + (\alpha - 1)\frac{1}{n}\sum_{i=1}^{n} \log |x_i| - \frac{1}{\sigma^\alpha \kappa^\alpha}\left(\kappa^{2\alpha}\bar{x}_\alpha^+ + \bar{x}_\alpha^-\right) \Big\}, \tag{72}$$

where

$$\bar{x}_\alpha^+ = \frac{1}{n}\sum_{i=1}^{n}(x_i^+)^\alpha, \quad \bar{x}_\alpha^- = \frac{1}{n}\sum_{i=1}^{n}(x_i^-)^\alpha, \quad \bar{x}_{\text{sign}} = \frac{1}{n}\sum_{i=1}^{n}\text{sign}(x_i),$$

and x_i^+ and x_i^- are given by (44) as before.

2.1.1 Fisher information matrix

If X has the $\mathcal{ADW}_\alpha(\sigma, \kappa)$ distribution with the vector-parameter $\gamma = (\gamma_1,\, \gamma_2)' = (\sigma,\, \kappa)'$ and density g given by (9), a straightforward calculation of the Fisher information matrix produces the following result:

$$I(\sigma,\, \kappa) = \begin{bmatrix} \frac{\alpha^2}{\sigma^2} & \frac{\alpha^2}{\sigma\kappa}\frac{\kappa^2-1}{\kappa^2+1} \\ \frac{\alpha^2}{\sigma\kappa}\frac{\kappa^2-1}{\kappa^2+1} & \frac{\alpha^2}{\kappa^2} + \frac{4}{(1+\kappa^2)^2} \end{bmatrix}. \tag{73}$$

2.1.2 Case 1: The value of σ is unknown

Using the result in (72), the following function needs to be maximized

$$Q(\sigma) = -\alpha \log \sigma - \frac{\kappa^\alpha}{\sigma^\alpha}\bar{x}_\alpha^+ - \frac{1}{\kappa^\alpha \sigma^\alpha}\bar{x}_\alpha^-.$$

A unique maximum likelihood estimator (MLE) of σ is obtained and presented with the following expression:

$$\hat{\sigma}_n = \left(\kappa^\alpha \bar{X}_\alpha^+ + \frac{1}{\kappa^\alpha}\bar{X}_\alpha^-\right)^{\frac{1}{\alpha}}. \tag{74}$$

.

Proposition 2.1 *Let $X_1, \ldots, X_n$ be i.i.d. r.v.'s from the $\mathcal{ADW}_\alpha(\sigma, \kappa)$ distribution with an unknown value of σ. The MLE of σ is given by (74) and is*

(i) consistent;

(ii) asymptotically normal, that is $\sqrt{n}(\hat{\sigma}_n - \sigma) \xrightarrow{d} N(0, \frac{\sigma^2}{\alpha^2})$;

(iii) asymptotically efficient, that is, the asymptotic variance σ^2/α^2 coincides with the reciprocal of the Fisher information $I(\sigma)$.

Proof. Write

$$\hat{\sigma}_n = h\left(\frac{1}{n}\sum_{i=1}^n W_i\right),$$

where

$$W_i = \kappa^\alpha (X_i^+)^\alpha + \frac{1}{\kappa^\alpha}(X_i^-)^\alpha = \begin{cases} \kappa^\alpha X_i^\alpha, & X_i \geq 0, \\ \frac{1}{\kappa^\alpha}(-X_i)^\alpha, & X_i < 0, \end{cases}$$

and $h(x) = x^{\frac{1}{\alpha}}, \ \ x \geq 0$, and note that the variables W_i are i.i.d. with mean $\mathbb{E}W_i = \sigma^\alpha$ and variance $\mathbb{V}\text{ar}W_i = \sigma^{2\alpha}$ (by the relations (52)-(53)). Thus, by the law of large numbers and the continuity of h, we have

$$\hat{\sigma}_n = h\left(\frac{1}{n}\sum_{i=1}^n W_i\right) \xrightarrow{d} h(\sigma^\alpha) = (\sigma^\alpha)^{\frac{1}{\alpha}} = \sigma,$$

which proves the consistency. By the central limit theorem,

$$n^{\frac{1}{2}}\left(\frac{1}{n}\sum_{i=1}^n W_i - \sigma^\alpha\right) \xrightarrow{d} N\left(0, \sigma^{2\alpha}\right),$$

where the right-hand-side denotes a normal variable with mean 0 and variance $\sigma^{2\alpha}$. Thus, by the continuity of h and standard large sample theory results (see Serfling [87]), we have

$$n^{\frac{1}{2}}\left(h\left(\frac{1}{n}\sum_{i=1}^n W_i\right) - h(\sigma^\alpha)\right) \xrightarrow{d} N(0, \eta^2), \tag{75}$$

where

$$\eta^2 = \left[h'(x)|_{x=\sigma^\alpha}\right]^2 \sigma^{2\alpha} = (\sigma^{1-\alpha}/\alpha)^2\sigma^{2\alpha} = \frac{\sigma^2}{\alpha^2}. \tag{76}$$

This proves Part (ii). The asymptotic efficiency is obtained by noting that the asymptotic variance σ^2/α^2 is the same as the reciprocal of the Fisher information $I(\sigma)$ given by the first entry in the Fisher information matrix (73).

2.1.3 Case 2: The value of κ is unknown

Using (72), the following function needs to be maximized:

$$Q(\kappa) = (1 + (\alpha - 1)\bar{x}_{\text{sign}}) \log \kappa - \log(1 + \kappa^2) - \frac{\kappa^\alpha}{\sigma^\alpha}\bar{x}_\alpha^+ - \frac{1}{\kappa^\alpha \sigma^\alpha}\bar{x}_\alpha^-. \tag{77}$$

The result is shown in the following Proposition.

Proposition 2.2 *If not both $\bar{x}_\alpha^+$ and $\bar{x}_\alpha^-$ are equal to zero, then there exists a unique $\hat{\kappa}_n \in (0, \infty)$ that maximizes the function $Q(\kappa)$ in (77). The value of $\hat{\kappa}_n$ is a unique positive solution of the equation*

$$(1 + (\alpha - 1)\bar{x}_{sign})(1 + \kappa^2)\kappa^\alpha - 2\kappa^{\alpha+2} + \frac{\alpha}{\sigma^\alpha}(1 + \kappa^2)(\bar{x}_\alpha^- - \bar{x}_\alpha^+ \kappa^{2\alpha}) = 0. \tag{78}$$

Proof. For simplicity, denote

$$A = \frac{1}{\sigma^\alpha}\bar{x}_\alpha^+ \quad B = \frac{1}{\sigma^\alpha}\bar{x}_\alpha^-, \text{ and } D = 1 + (\alpha - 1)\bar{x}_{\text{sign}}.$$

Several cases will be considered.

Case 1: $A > 0, B = 0$. Here, all sample values are positive, in which case $D = \alpha$. We need to maximize the function

$$Q(\kappa) = \alpha \log \kappa - \log(1 + \kappa^2) - A\kappa^\alpha \tag{79}$$

with respect to $\kappa \in (0, \infty)$. The derivative of Q is

$$u(\kappa) = \frac{dQ}{d\kappa} = \frac{1}{\kappa(1 + \kappa^2)}(u_1(\kappa) - u_2(\kappa)),$$

where

$$u_1(\kappa) = \alpha - (2 - \alpha)\kappa^2 \text{ and } u_2(\kappa) = A\alpha\kappa^\alpha(1 + \kappa^2).$$

Consider the above functions on the interval $(0, \infty)$. Note that the derivative u is continuous, positive when $\kappa \to 0^+$, and negative when $\kappa \to \infty$. Consequently, there is at least one solution of the equation $u(\kappa) = 0$, or equivalently, $u_1(\kappa) = u_2(\kappa)$. We claim that there is exactly one solution. Indeed, when $0 < \alpha < 2$, then u_1 is strictly decreasing while u_2 is strictly increasing, when $\alpha = 2$, we have $u_1(\kappa) = 2$ for each

κ (with u_2 being strictly increasing), and when $\alpha > 2$, both u_1 and u_2 are strictly increasing and concave. We conclude that there is a unique value $\hat{\kappa} \in (0, \infty)$ that maximizes the function Q, and can be found by solving the equation $u_1(\kappa) - u_2(\kappa) = 0$, which coincides with (78) when $B = 0$. This concludes Case 1.

Case 2: $A = 0, B > 0$. Here, all sample values are negative, and $D = 2 - \alpha$. We need to maximize the function

$$Q(\kappa) = (2 - \alpha) \log \kappa - \log(1 + \kappa^2) - \frac{B}{\kappa^\alpha}.$$

Note that we have

$$Q(1/\kappa) = \alpha \log \kappa - \log(1 + \kappa^2) - B\kappa^\alpha,$$

which is the same function as (79) of Case 1 with $A > 0$ replaced by $B > 0$. Thus, this case follows from the previous one.

Case 3: $A > 0$ *and* $B > 0$. Write the derivative of Q in (77) as

$$v(\kappa) = \frac{dQ(\kappa)}{d\kappa} = \frac{\kappa^\alpha (D - \kappa^2 (2 - D)) + \alpha(1 + \kappa^2)(B - A\kappa^{2\alpha})}{(1 + \kappa^2)\kappa^{\alpha+1}}. \tag{80}$$

Note that the function v is continuous on $(0, \infty)$, positive when $\kappa \to 0^+$, and negative when $\kappa \to \infty$. Consequently, v has at least one positive zero. Below, we show that v has exactly one positive zero, which corresponds to the maximum value of Q. To see this, first note that a zero of v satisfies the equation

$$\kappa^\alpha (D - \kappa^2 (2 - D)) + \alpha(1 + \kappa^2)(B - A\kappa^{2\alpha}) = 0. \tag{81}$$

Next, denote

$$y = \kappa^\alpha, \quad x = \kappa^2,$$

and observe that if $\kappa > 0$ satisfies (81), then x and y satisfies the system

$$\begin{cases} y(D - x(2 - D)) + \alpha(1 + x)(B - Ay^2) = 0 \\ y = x^{\frac{\alpha}{2}}. \end{cases} \tag{82}$$

We claim that the system (82) does not admit more than one solution in the region $x, y > 0$. Indeed, the first equation is quadratic in y and can be solved easily for y in terms of x leading to

$$y = \frac{1}{2A\alpha} \left(\sqrt{\left(\frac{(2 - D)x - D}{x + 1} \right)^2 + 4AB\alpha^2} - \frac{(2 - D)x - D}{x + 1} \right) = h(u(x)),$$

where

$$u(x) = \frac{(2 - D)x - D}{x + 1}, \quad x \in [0, \infty),$$

and

$$h(u) = \frac{1}{2A\alpha}(\sqrt{u^2 + 4AB\alpha^2} - u) = \frac{2B\alpha}{\sqrt{u^2 + 4AB\alpha^2} + u}, \quad u \in (-\infty, \infty).$$

Note that the function u is increasing on $(0, \infty)$, while the function h is decreasing on $(-\infty, \infty)$. Thus, the function $v(x) = h(u(x))$ is decreasing on $(0, \infty)$, so that the system of equations (82), which can be written as

$$\begin{cases} y = v(x) \\ y = x^{\frac{\alpha}{2}}, \end{cases}$$

has at most one solution in the region $x, y > 0$. This concludes Case 3, and the result follows.

The properties of the MLE are presented below.

Proposition 2.3 *Let $X_1, \ldots, X_n$ be i.i.d. variables from an $\mathcal{ADW}_\alpha(\sigma, \kappa)$ distribution where the values of α and σ are known. Then the MLE of κ, which is the unique solution $\hat{\kappa}_n$ of the equation (78), is*

(i) consistent;

(ii) asymptotically normal, that is, $\sqrt{n}(\hat{\kappa}_n - \kappa) \stackrel{d}{\to} N(0, \sigma_\kappa^2)$, where

$$\sigma_\kappa^2 = \frac{\kappa^2(1+\kappa^2)^2}{\alpha^2(1+\kappa^2)^2 + 4\kappa^2}; \tag{83}$$

(iii) asymptotically efficient, that is, the asymptotic variance (83) coincides with the reciprocal of the Fisher information $I(\kappa)$.

To prove this result, we need the following lemma, which can be established by a straightforward albeit lengthy algebra.

Lemma 2.1 *Let X have an $\mathcal{ADW}_\alpha(\sigma, \kappa)$ distribution, and let*

$$W = [W_1, W_2, W_3]' = [(X^+)^\alpha, (X^-)^\alpha, sign(X)]'.$$

Then the mean vector and the covariance matrix of W are

$$\mathbb{E}W' = \left[\frac{\sigma^\alpha}{\kappa^\alpha(1+\kappa^2)}, \frac{\sigma^\alpha\kappa^{\alpha+2}}{1+\kappa^2}, \frac{1-\kappa^2}{1+\kappa^2}\right]' \tag{84}$$

and

$$\mathbb{C}ov(W) = \Sigma_W = \frac{1}{(1+\kappa^2)^2}\begin{bmatrix} \frac{\sigma^{2\alpha}(1+2\kappa^2)}{\kappa^{2\alpha}} & -\kappa^2\sigma^{2\alpha} & \frac{2\kappa^2\sigma^\alpha}{\kappa^\alpha} \\ -\kappa^2\sigma^{2\alpha} & \sigma^{2\alpha}\kappa^{2\alpha+2}(2+\kappa^2) & -2\kappa^{\alpha+2}\sigma^\alpha \\ \frac{2\kappa^2\sigma^\alpha}{\kappa^\alpha} & -2\kappa^{\alpha+2}\sigma^\alpha & 4\kappa^2 \end{bmatrix}, \tag{85}$$

respectively.

Proof of Proposition 2.3. Since the MLE $\hat{\kappa}_n$ is a unique solution of (78), it can be written as

$$\hat{\kappa}_n = H(\bar{x}_\alpha^+, \bar{x}_\alpha^-, \bar{x}_{\text{sign}}),$$

where $H(\cdot, \cdot, \cdot)$ is a continuous and differentiable function satisfying the equation

$$F(\bar{x}_\alpha^+, \bar{x}_\alpha^-, \bar{x}_{\text{sign}}, H(\bar{x}_\alpha^+, \bar{x}_\alpha^-, \bar{x}_{\text{sign}})) = 0$$

with

$$F(y_1, y_2, y_3, y_4) = [1 + (\alpha - 1)y_3]y_4^\alpha(1 + y_4^2) - 2y_4^{\alpha+2} + \frac{\alpha}{\sigma^\alpha}(1 + y_4^2)(y_2 - y_1 y_4^{2\alpha}).$$

(i) To establish the consistency of the MLE, note that by the law of large numbers, we have

$$(\bar{X}_\alpha^+, \bar{X}_\alpha^-, \bar{X}_{\text{sign}})' \xrightarrow{d} (\mu_1, \mu_2, \mu_3)' = \left[\frac{\sigma^\alpha}{\kappa^\alpha(1 + \kappa^2)}, \frac{\sigma^\alpha \kappa^{2+\alpha}}{1 + \kappa^2}, \frac{1 - \kappa^2}{1 + \kappa^2}\right]',$$

see Lemma 2.1. Thus, by the continuity of H, we obtain

$$\hat{\kappa}_n = H(\bar{X}_\alpha^+, \bar{X}_\alpha^-, \bar{X}_{\text{sign}})' \xrightarrow{d} H(\mu_1, \mu_2, \mu_3) = \kappa,$$

since the quantities

$$y_1 = \mu_1 = \frac{\sigma^\alpha}{\kappa^\alpha(1 + \kappa^2)}, \; y_2 = \mu_2 = \frac{\sigma^\alpha \kappa^{2+\alpha}}{1 + \kappa^2}, \; y_3 = \frac{1 - \kappa^2}{1 + \kappa^2}, \; y_4 = \kappa \tag{86}$$

satisfy the equation $F(y_1, y_2, y_3, y_4) = 0$.

(ii) According to the central limit theorem, we have

$$\sqrt{n}[(\bar{X}_\alpha^+, \; \bar{X}_\alpha^-, \; \bar{X}_{\text{sign}})' - (\mu_1, \; \mu_2, \; \mu_3)'] \to N(0, \Sigma_W),$$

where the right-hand-side denotes a bivariate normal variable with mean zero and variance-covariance matrix Σ_W given by (85). Now, by standard large sample theory results (see Serfling [87]) it follows that as $n \to \infty$, we have

$$\sqrt{n}[H(\bar{X}_\alpha^+, \; \bar{X}_\alpha^-, \; \bar{X}_{\text{sign}})' - H(\mu_1, \; \mu_2, \; \mu_3)] \to N(0, \Omega),$$

where $\Omega = D\Sigma_W D'$ and D is the vector of partial derivatives

$$D = \left[\frac{\partial H}{\partial y_1}, \frac{\partial H}{\partial y_2}, \frac{\partial H}{\partial y_3}\right]_{(y_1, y_2, y_3) = (\mu_1, \mu_2, \mu_3)}. \tag{87}$$

Since the function H satisfies the equation

$$F(y_1, y_2, y_3, H(y_1, y_2, y_3)) = 0,$$

we have

$$\frac{\partial}{\partial y_1}H(y_1,y_2,y_3)=\frac{\frac{\partial}{\partial y_1}F(y_1,y_2,y_3,y_4)|_{y_4=H(y_1,y_2,y_3)}}{\frac{\partial}{\partial y_4}F(y_1,y_2,y_3,y_4)|_{y_4=H(y_1,y_2,y_3)}}, \tag{88}$$

$$\frac{\partial}{\partial y_2}H(y_1,y_2,y_3)=\frac{\frac{\partial}{\partial y_2}F(y_1,y_2,y_3,y_4)|_{y_4=H(y_1,y_2,y_3)}}{\frac{\partial}{\partial y_4}F(y_1,y_2,y_3,y_4)|_{y_4=H(y_1,y_2,3_3)}}, \tag{89}$$

$$\frac{\partial}{\partial y_3}H(y_1,y_2,y_3)=\frac{\frac{\partial}{\partial y_3}F(y_1,y_2,y_3,y_4)|_{y_4=H(y_1,y_2,y_3)}}{\frac{\partial}{\partial y_4}F(y_1,y_2,y_3,y_4)|_{y_4=H(y_1,y_2,3_3)}}, \tag{90}$$

Routine calculations produce

$$\frac{\partial}{\partial y_1}F(y_1,y_2,y_3,y_4)=-\frac{\alpha}{\sigma^\alpha}y_4^{2\alpha}(1+y_4^2),$$

$$\frac{\partial}{\partial y_2}F(y_1,y_2,y_3,y_4)=\frac{\alpha}{\sigma^\alpha}(1+y_4^2),$$

$$\frac{\partial}{\partial y_3}F(y_1,y_2,y_3,y_4)=(\alpha-1)(1+y_4^2)y_4^\alpha,$$

$$\frac{\partial}{\partial y_4}F(y_1,y_2,y_3,y_4)=[1+(\alpha-1)y_3][\alpha y_4^{\alpha-1}+(\alpha+2)y_4^{\alpha+1}]-2(2+\alpha)y_4^{1+\alpha}+$$

$$+\frac{\alpha}{\sigma^\alpha}[-2\alpha y_1y_4^{2\alpha-1}+2y_2y_4-(2\alpha+2)y_1y_4^{2\alpha+1}].$$

Substituting (86) into the above derivatives, we obtain

$$\frac{\partial}{\partial y_1}F(y_1,y_2,y_3,y_4)|_{(y_1,y_2,y_3,y_4)=(\mu_1,\mu_2,\mu_3,H(\mu_1,\mu_2,\mu_3))}=-\frac{\alpha}{\sigma^\alpha}\kappa^{2\alpha}(1+\kappa^2),$$

$$\frac{\partial}{\partial y_2}F(y_1,y_2,y_3,y_4)|_{(y_1,y_2,y_3,y_4)=(\mu_1,\mu_2,\mu_3,H(\mu_1,\mu_2,\mu_3))}=\frac{\alpha}{\sigma^\alpha}(1+\kappa^2),$$

$$\frac{\partial}{\partial y_3}F(y_1,y_2,y_3,y_4)|_{(y_1,y_2,y_3,y_4)=(\mu_1,\mu_2,\mu_3,H(\mu_1,\mu_2,\mu_3))}=(\alpha-1)(1+\kappa^2)\kappa^\alpha,$$

$$\frac{\partial}{\partial y_4}F(y_1,y_2,y_3,y_4)|_{(y_1,y_2,y_3,y_4)=(\mu_1,\mu_2,\mu_3,H(\mu_1,\mu_2,\mu_3))}=-\frac{\kappa^{\alpha-1}}{1+\kappa^2}[4\kappa^2+\alpha^2(1+\kappa^2)^2].$$

Consequently, the vector of partial derivatives (87) takes the form

$$D=-\frac{(1+\kappa^2)^2}{\kappa^{\alpha-1}(4\kappa^2+\alpha^2(1+\kappa^2)^2)}\left[-\frac{\alpha}{\sigma^\alpha}\kappa^{2\alpha},\ \frac{\alpha}{\sigma^\alpha},\ (\alpha-1)\kappa^\alpha\right].$$

After some algebra, we find that the product $\Omega=D\Sigma_W D'$ coincides with (83). This concludes the proof of asymptotic normality.

(iii) To establish the asymptotic efficiency, note that the asymptotic variance is the reciprocal of the Fisher information (see Fisher information matrix (73)).

2.1.4 Case 3: The values of σ and κ are unknown

Here we need to maximize the function

$$Q(\kappa,\sigma) = (1+(\alpha-1)\bar{x}_{\text{sign}})\log\kappa - \log(1+\kappa^2) - \alpha\log\sigma - \frac{\kappa^\alpha}{\sigma^\alpha}\bar{x}_\alpha^+ - \frac{1}{\kappa^\alpha\sigma^\alpha}\bar{x}_\alpha^-, \quad (91)$$

where $\bar{x}_\alpha^+$, $\bar{x}_\alpha^-$, and $\bar{x}_{\text{sign}}$ are as before. Note that for each fixed $\kappa > 0$, the maximum of Q occurs at the point $(\kappa, \sigma(\kappa))$, where $\sigma(\kappa)$ is the MLE of σ, given by (74). Thus, for each $\kappa, \sigma > 0$ we have

$$Q(\kappa,\sigma) \le Q(\kappa,\sigma(\kappa)) = (1+(\alpha-1)\bar{x}_{\text{sign}})\log\kappa - \log(1+\kappa^2) - \log\left[\kappa^\alpha\bar{x}_\alpha^+ + \frac{\bar{x}_\alpha^-}{\kappa^\alpha}\right] - 1. \quad (92)$$

Consider first the case when all sample values are positive ($x_i^+ = x_i$ and $x_i^- = 0$ for each i). In this case we have $\bar{x}_\alpha^+ > 0$, $\bar{x}_\alpha^- = 0$, and $\bar{x}_{\text{sign}} = 1$, and the right-hand-side of (92) takes the form

$$Q(\kappa,\sigma(\kappa)) = -\log(1+\kappa^2) - \log\bar{x}_\alpha^+ - 1.$$

Since this function is decreasing in κ, the least upper bound of $Q(\kappa,\sigma)$ is attained when $\kappa = 0$ and $\sigma = \sigma(0) = 0$. Although these values of the parameters are not visible, as $\kappa \to 0^+$ and $\sigma(\kappa) = \kappa(\bar{x}_\alpha^+)^{\frac{1}{\alpha}} \to 0^+$, the $\mathcal{ADW}_\alpha(\sigma(\kappa),\kappa)$ distribution converges to the classical Weibull distribution with scale parameter $(\bar{x}_\alpha^+)^{\frac{1}{\alpha}}$ (see the discussion of special cases following Definition 1.1). Intuitively, it makes sense to conclude that the underlying distribution is concentrated on $(0,\infty)$ if all sample values are positive.

Next, consider the case when all sample values are negative, so that $\bar{x}_\alpha^+ = 0$, $\bar{x}_\alpha^- > 0$, and $\bar{x}_{\text{sign}} = -1$. Here, the right-hand-side of (92) takes the form

$$Q(\kappa,\sigma(\kappa)) = 2\log\kappa - \log(1+\kappa^2) - \log\bar{x}_\alpha^- - 1,$$

which is an increasing function of κ. Consequently, the maximum value of $Q(\kappa,\sigma)$ occurs in the limit when $\kappa \to \infty$ and $\sigma(\kappa) = (\bar{x}_\alpha^-)^{\frac{1}{\alpha}}/\kappa \to 0$. This limiting distribution corresponds to a random variable $-X$, where X has a classical Weibull distribution with scale parameter $(\bar{x}_\alpha^-)^{\frac{1}{\alpha}}$. Again, it is reasonable that the distribution is concentrated on the negative half line if all sample values are negative. Finally, if not all sample values are positive (or negative), we have the following result.

Proposition 2.4 *Let $X_1,\ldots,X_n$ be an i.i.d. variables from the $\mathcal{ADW}_\alpha(\sigma,\kappa)$ distribution with known α, and suppose that*

$$\bar{X}_\alpha^+ = \frac{1}{n}\sum_{i=1}^n (X_i^+)^\alpha > 0 \text{ and } \bar{X}_\alpha^- = \frac{1}{n}\sum_{i=1}^n (X_i^-)^\alpha > 0.$$

Then, there exist unique MLE's of κ and σ: $\hat{\kappa}_n$ is the unique positive solution of the equation

$$[1+(\alpha-1)\bar{X}_{sign}]+\alpha-\frac{2\kappa^2}{1+\kappa^2}-\frac{2\alpha\bar{X}_\alpha^+\kappa^{2\alpha}}{\bar{X}_\alpha^+\kappa^{2\alpha}+\bar{X}_\alpha^-}=0, \tag{93}$$

where $\bar{X}_{sign}=\frac{1}{n}\sum_{i=1}^n sign(X_i)$, and

$$\hat{\sigma}_n=\left(\hat{\kappa}_n^\alpha\bar{X}_\alpha^+ + \frac{1}{\hat{\kappa}_n^\alpha}\bar{X}_\alpha^-\right)^{\frac{1}{\alpha}}. \tag{94}$$

Proof. We need to maximize the function $u(\kappa)=Q(\kappa,\sigma(\kappa))$ given by the right-hand-side of (92). Write

$$\frac{\partial}{\partial\kappa}u(\kappa)=\frac{1}{\kappa}\left(h_1(\kappa)-h_2(\kappa)\right), \tag{95}$$

where

$$h_1(\kappa)=D+\alpha-\frac{2\kappa^2}{1+\kappa^2},\ \ h_2(\kappa)=\frac{2\alpha\bar{X}_\alpha^+\kappa^{2\alpha}}{\bar{X}_\alpha^+\kappa^{2\alpha}+\bar{X}_\alpha^-},\ \ D=1+(\alpha-1)\bar{X}_{\text{sign}}.$$

Observe that the function h_1 is monotonically decreasing on the interval $(0,\infty)$ with $h_1(0)=D+\alpha>0$ and $h_1(\infty)=D+\alpha-2$. On the other hand, the function h_2 is monotonically increasing on the interval $(0,\infty)$, with $h_2(0)=0$ and $h_2(\infty)=2\alpha> D+\alpha-2$. Consequently, there exist a unique point $\hat{\kappa}_n\in(0,\infty)$, such that the above derivative is negative for $\kappa>\hat{\kappa}_n$ and positive for $0<\kappa<\hat{\kappa}_n$, showing that the function u attains a unique maximum value on the interval $(0,\infty)$. The value that maximizes u is obtained by solving the equation $u'(\kappa)=0$, which is the same as (93). This concludes the proof.

Remark. Note that when $\alpha=1$, which corresponds to skew Laplace distribution (see Kotz et al. [49]), the MLE of κ takes an explicit form: $\hat{\kappa}_n=\sqrt[4]{\bar{X}_\alpha^-/\bar{X}_\alpha^+}$.

We skip the technical derivation of the following result, as it can be established in almost the same way as Proposition 2.3.

Proposition 2.5 *Let $X_1,\ldots,X_n$ be i.i.d. variables from the $\mathcal{ADW}_\alpha(\sigma,\kappa)$ distribution with unknown values of σ and κ. Then, the MLE's of σ and κ given in Proposition 2.4 are*

(i) consistent;

(ii) asymptotically bivariate normal, with the asymptotic covariance matrix

$$\Sigma_{MLE}=\frac{\sigma^2}{4(\alpha^2+1)}\begin{bmatrix}\frac{(1+\kappa^2)^2}{\kappa^2}+\frac{4}{\alpha^2} & \frac{1-\kappa^4}{\sigma\kappa}\\ \frac{1-\kappa^4}{\sigma\kappa} & \frac{(1+\kappa^2)^2}{\sigma^2}\end{bmatrix}, \tag{96}$$

(iii) asymptotically efficient, that is, the above asymptotic covariance matrix coincides with the inverse of the Fisher information matrix.

2.2 Asymmetric double Weibull Distribution of Type II

Let $X_1,\ldots,X_n$ be an i.i.d. random sample from $\mathcal{ADW}^*_\alpha(\sigma,\kappa)$ distribution with density (10), and let $x_1,\ldots,x_n$ be their particular realization. The log-likelihood function then becomes

$$\log L^*(\alpha,\sigma,\kappa) = n\left\{\log\alpha + \log\frac{\kappa}{1+\kappa^2} - \log\sigma + (\alpha-1)\frac{1}{n}\sum_{i=1}^{n}\log|x_i| - \frac{1}{\sigma}\left(\kappa\bar{x}^+_\alpha + \frac{\bar{x}^-_\alpha}{\kappa}\right)\right\}, \tag{97}$$

where $\bar{x}^+_\alpha$ and $\bar{x}^-_\alpha$ are defined as before. Again, here our main focus is the estimation of σ and κ assuming that α is known. The results presented below can be obtained in almost the same way as those in the previous section. Moreover, since the transformation $Y_i = X_i^{<\alpha>}$, $i=1,\ldots,n$, yields a random sample from skew Laplace distribution (45), we can apply results already available for this case (see Kotz et al. [50]). For this reason we shall skip computational details, and refer to Jurić [41] for technical details.

2.2.1 Fisher information matrix

If X has the $\mathcal{ADW}^*_\alpha(\sigma,\kappa)$ distribution with the vector-parameter $\gamma=(\gamma_1,\,\gamma_2)'=(\sigma,\,\kappa)'$ and density g given by (10), routine calculations produce the following result:

$$I(\sigma,\kappa) = \begin{bmatrix} \frac{1}{\sigma^2} & \frac{\kappa^2-1}{\sigma\kappa(1+\kappa^2)} \\ \frac{\kappa^2-1}{\sigma\kappa(1+\kappa^2)} & \frac{\kappa^4+6\kappa^2+1}{\kappa^2(1+\kappa^2)^2} \end{bmatrix}, \tag{98}$$

see Jurić [41] for details.

2.2.2 Case 1: The value of σ is unknown

Here, we need to maximize the function

$$Q(\sigma) = -\log\sigma - \frac{1}{\sigma}\left(\kappa\bar{x}^+_\alpha + \frac{\bar{x}^-_\alpha}{\kappa}\right).$$

It is easy to see that

$$\hat{\sigma}_n = \kappa\bar{X}^+_\alpha + \frac{1}{\kappa}\bar{X}^-_\alpha. \tag{99}$$

is the unique MLE of σ. The following result, which can be proved in the same way as Proposition 2.1, presents the asymptotic behavior of $\hat{\sigma}_n$.

Proposition 2.6 *Let* $X_1,\ldots,X_n$ *be i.i.d. r.v.'s from the* $\mathcal{ADW}^*_\alpha(\sigma,\kappa)$ *distribution with an unknown value of* σ*. The MLE of* σ *is given by (99) and is*

(i) consistent;

(ii) asymptotically normal, that is $\sqrt{n}(\hat{\sigma}_n - \sigma) \xrightarrow{d} N(0,\sigma^2)$*;*

(iii) asymptotically efficient, that is the asymptotic variance σ^2 *coincides with the reciprocal of the Fisher information* $I(\sigma)$*.*

2.2.3 Case 2: The value of κ is unknown

In view of (97), we need to maximize the function

$$Q(\kappa) = \log\kappa - \log(1+\kappa^2) - \frac{\kappa}{\sigma}\bar{x}^+_\alpha - \frac{1}{\kappa\sigma}\bar{x}^-_\alpha. \tag{100}$$

This is essentially the same function as that in the skew Laplace case (see Kotz et al. [49], Section 3.5.1.3), so the Laplace results imply that there is a unique positive value $\hat{\kappa}_n$ that maximizes the function Q, provided that not both $\bar{x}^+_\alpha$ and $\bar{x}^-_\alpha$ are equal to zero. Setting the derivative of Q to zero, we find that the MLE of κ is the unique positive solution of the equation

$$\frac{1}{\kappa} - \frac{2\kappa}{1+\kappa^2} + \frac{1}{\kappa^2}\frac{\bar{x}^-_\alpha}{\sigma} - \frac{\bar{x}^+_\alpha}{\sigma} = 0. \tag{101}$$

The properties of the MLE, which essentially follow from the Laplace case, are presented below.

Proposition 2.7 *Let* $X_1,\ldots,X_n$ *be i.i.d. variables from an* $\mathcal{ADW}^*_\alpha(\sigma,\kappa)$ *distribution where the values of* α *and* σ *are known. Then the MLE of* κ*, which is the unique solution* $\hat{\kappa}_n$ *of the equation (101), is*

(i) consistent;

(ii) asymptotically normal, that is, $\sqrt{n}(\hat{\kappa}_n - \kappa) \xrightarrow{d} N(0,\, \sigma^2_\kappa)$*, where*

$$\sigma^2_\kappa = \frac{\kappa^2(1+\kappa^2)^2}{(1+\kappa^2)^2 + 4\kappa^2}; \tag{102}$$

(iii) asymptotically efficient, that is, the asymptotic variance (102) coincides with the reciprocal of the Fisher information $I(\kappa)$*.*

2.2.4 Case 3: The values of σ and κ are unknown

The analysis of this case parallels the analogous case connected with the skew double Weibull distribution of type I. For any fixed $\kappa > 0$ the log-likelihood function (97) is maximized by the MLE of $\hat{\sigma}_n$ given by (99). When we substitute this value into the log-likelihood function, we are left with maximizing the resulting expression,

$$Q(\kappa) = 2\log\kappa - \log(\kappa^2\bar{x}_\alpha^+ + \bar{x}_\alpha^-) - \log(1+\kappa^2),$$

with respect to κ. This is essentially the skew Laplace case (on transformed data) discussed in Kotz et al. (see [50]), so we shall skip the derivation of the results below (see Jurić [41] for details). If all sample values are positive (so that $\bar{x}_\alpha^- = 0$), the above function is decreasing, so the maximum occurs at $\kappa = 0$ (and $\sigma = 0$), corresponding to a one sided Weibull distribution with scale parameter $(\bar{x}_\alpha^+)^{\frac{1}{\alpha}}$ (as in the case of Type I distribution). Similarly, when all sample values are negative (so that $\bar{x}_\alpha^+ = 0$), the function Q is increasing in κ, with the maximum occurring at $\kappa = \infty$ (and $\sigma = 0$). This corresponds to the variable $-X$, where X has a classical Weibull distribution with scale parameter $(\bar{x}_\alpha^-)^{\frac{1}{\alpha}}$. If not all sample values are positive (or negative), then we have the following result (see Jurić [41]).

Proposition 2.8 *Let $X_1,\ldots,X_n$ be i.i.d. variables from the $\mathcal{ADW}_\alpha^*(\sigma,\kappa)$ distribution with unknown values of σ and κ. Then, if $\bar{X}_\alpha^+ > 0$ and $\bar{X}_\alpha^- > 0$, the MLE's of σ and κ are given by*

$$\hat{\sigma}_n = \left(\frac{\bar{X}_\alpha^-}{\bar{X}_\alpha^+}\right)^{\frac{1}{4}} \bar{X}_\alpha^+ + \left(\frac{\bar{X}_\alpha^+}{\bar{X}_\alpha^-}\right)^{\frac{1}{4}} \bar{X}_\alpha^- \quad \text{and} \quad \hat{\kappa}_n = \sqrt[4]{\frac{\bar{X}_\alpha^-}{\bar{X}_\alpha^+}}, \tag{103}$$

respectively. Moreover, the MLE's are

(i) consistent;

(ii) asymptotically bivariate normal, with the asymptotic covariance matrix

$$\Sigma_{MLE} = \begin{bmatrix} \frac{\sigma^2(\kappa^4+6\kappa^2+1)}{8\kappa^2} & \frac{\sigma(1-\kappa^4)}{8\kappa} \\ \frac{\sigma(1-\kappa^4)}{8\kappa} & \frac{(1+\kappa^2)^2}{8} \end{bmatrix};$$

(iii) asymptotically efficient, that is, the asymptotic covariance matrix coincides with the inverse of the Fisher information matrix.

Remark. Observe that unlike the similar case of skew distribution of Type I, here both estimators are given explicitly.

2.3 Estimation of σ, κ and α

For both Type I and II distributions, the estimation of α requires a numerical approach, as is the case with the classical Weibull distribution. For example, for Type II distribution, substitute the MLE's of σ and κ given in Proposition 2.8 into the log-likelihood function, which (after some algebra) leads to the problem of maximizing the function

$$Q(\alpha) = \log \frac{\alpha \hat{\kappa}(\alpha)}{\hat{\sigma}(\alpha)(1 + \hat{\kappa}(\alpha)^2)} + (\alpha - 1)\frac{1}{n}\sum_{i=1}^{n} \log |x_i|$$

with respect to $\alpha > 0$. Here, $\hat{\kappa}(\alpha)$ and $\hat{\sigma}(\alpha)$ are the MLE's given by (103). When optimizing the function Q numerically, one usually restricts the range of the α's to a finite interval, typically including $\alpha = 1$ corresponding to the Laplace case.

3 Application - the univariate case

In this chapter we present a skew double Weibull model for currency exchange rates. The results below closely follow Jurić and Kozubowski [43].

3.1 Modeling currency exchange rates (conditional distribution of the changes)

Many different distributions have been proposed in the past to model currency exchange rates, including stable Paretian laws (see Westerfield [100], McFarland et al. [72, 73], So [89], Koedjik et al. [47] and Nolan [77]), Student-t distribution (see Boothe and Glassman [8] and Koedjik et al. [47]), mixture of normals (see Boothe and Glassman [8] and Tucker and Pond [94]), asymmetric Laplace distribution (see Kozubowski and Podgórski [52]), and exponential power distribution (see Ayebo and Kozubowski [6]). While there is still no general consensus regarding the best theoretical model, Chenayo et al. (see [9]) found the fit of a double Weibull model to be the best.

Following the proposals of Chenyao et al. (see [9]), Hürliman (see [34]), and Mittnik and Rachev (see [74]), who reported excellent results with the (double) Weibull distribution, we propose the *asymmetric* double Weibull distribution $\mathcal{ADW}^*$ to model currency exchange rates. We fit this distribution to our data, and compare its performance with that of the normal, asymmetric Laplace (AL), and exponential power (EP) distributions.

Our data are daily currency exchange rates, quoted in U.K. pounds, for fifteen currencies reported for the period of January 2, 1980 to May 21, 1996. These data were used before in Nolan (see [77]), Kozubowski and Podgórski (see [52]), and Ayebo and Kozubowski (see [6]). Following a common practice, we converted the daily data P_i to $X_i = \log(P_{i+1}/P_i)$ (the logarithmic return), resulting in 4274 values for each currency.

The models used include the asymmetry parameter κ (see Table 1). The estimation shows that κ is close to 1 in all cases which does not imply asymmetry for the data set used. However, this can be interpreted as statistical evidence that it may be necessary to model asymmetry. This point is also considered in Kozubowski and Panorska (see [60]).

Currency	α	σ	κ
Australia	1.0931	0.0037	0.9994
Austria	1.1585	0.0017	1.0234
Belgium	1.1373	0.0023	1.0117
Canada	1.1490	0.0024	1.0066
Denmark	1.1384	0.0018	1.0148
France	1.0789	0.0023	1.0073
Germany	1.2079	0.0012	1.0276
Italy	1.0467	0.0029	0.9916
Japan	1.2322	0.0017	1.0393
Netherlands	1.1979	0.0013	1.0255
Norway	1.1286	0.0018	1.0034
Spain	1.1021	0.0022	0.9916
Sweden	1.0673	0.0026	0.9945
Switzerland	1.2462	0.0012	1.0294
U.S.	1.1363	0.0026	1.0111

Table 1: Estimated values of α, σ and κ of the fitted $\mathcal{ADW}^*_\alpha(\sigma, \kappa)$ distributions. Note: reprinted from [43]

We use maximum likelihood estimators presented in Chapter 2 to fit the skew Weibull model to the data. All zero returns have been excluded from the data set, since the likelihood function takes the logarithm of the data values. In effect, we model the *conditional distribution* of the changes, given that a non-zero change has occurred. This approach is different from that of prior studies, where stochastic models were fitted to the entire data, including the zero values (which are about 5% of the data in our case). The results of estimation for all fifteen currencies are presented in Table 1.

To compare the fits of competing models, we consider the standard Kolmogorov-Smirnov (K-S) distance between the data and the model distributions.

Fit	Australia	Austria	Belgium	Canada	Denmark
Normal	0.064	0.055	0.073	0.042	0.057
AL	0.028	0.049	0.060	0.031	0.048
EP	0.022	0.045	0.056	0.016	0.042
$\mathcal{ADW}^*$	**0.017**	**0.035**	**0.045**	**0.011**	**0.031**
Fit	**France**	**Germany**	**Italy**	**Japan**	**Netherlands**
Normal	0.075	0.066	0.070	0.049	0.065
AL	0.043	0.077	0.030	0.062	0.071
EP	0.043	0.071	0.029	0.052	0.063
$\mathcal{ADW}^*$	**0.038**	**0.055**	**0.025**	**0.038**	**0.048**
Fit	**Norway**	**Spain**	**Sweden**	**Switzerland**	**US**
Normal	0.058	0.064	0.093	0.055	0.046
AL	0.036	0.038	0.027	0.075	0.028
EP	0.030	0.035	0.027	0.067	0.019
$\mathcal{ADW}^*$	**0.021**	**0.026**	**0.022**	**0.049**	**0.016**

Table 2: Kolmogorov-Smirnov distances between the data and the four models: normal, asymmetric Laplace, skew exponential power and asymmetric double Weibull ($\mathcal{ADW}^*$). Note: reprinted from [43]

The values of this statistic are listed in Table 2. As we can see from the table, the normal distribution provides a rather poor fit, as expected, while the asymmetric double Weibull model works best. Note that, similarly to the case of modeling the entire data set (as reported in Ayebo and Kozubowski [6]), the skew EP model provides only a very slight improvement over the AL distribution. Overall, the skew Weibull model provides a substantial improvement over both, the AL and the skew EP models.

3.2 Modeling the unconditional distribution of the currency exchange rates

In our approach, the variable $X \sim \mathcal{ADW}^*_\alpha(\sigma, \kappa)$ represents the (logarithmic) change of the exchange rate, provided that one has actually occurred. Thus, the unconditional distribution can be written as the variable $Y = IX$, where I is a Bernoulli variable with parameter p, independent of X:

$$I = \begin{cases} 0, & \text{with prob. } 1-p \\ 1, & \text{with prob. } p \end{cases} \quad \text{and } Y = \begin{cases} 0, & \text{with prob. } 1-p \\ X, & \text{with prob. } p. \end{cases} \tag{104}$$

Consequently, the distribution of Y contains an atom at zero, and can be calculated in the following way:
(i) For $x < 0$,

$$P(Y \leq x) = P(IX \leq x) = P(I = 1 \text{ and } X \leq x) = pF(x). \tag{105}$$

(ii) For $x \geq 0$,

$$\begin{aligned} P(Y \leq x) &= P(IX \leq x) = P(I = 0 \text{ or } (I = 1 \text{ and } X \leq x)) \\ &= P(I = 0) + P(I = 1)P(X \leq x) = P(I = 0) + P(I = 1)F(x) \\ &= 1 - p + pF(x). \end{aligned} \tag{106}$$

Thus, the cumulative distribution function of Y is of the form:

$$F_Y(x) = \begin{cases} pF(x), & x < 0 \\ 1 - p + pF(x), & x \geq 0, \end{cases} \tag{107}$$

where F is the c.d.f. of X. In practice, the parameter p can be estimated as the proportion of non-zero returns in the data set, thus the probability that the exchange rate does change. When change occurs it is generated from the Weibull distribution. To illustrate, we consider the case of Australian dollar, where we obtain $\hat{p} = 4094/4274 = 0.9578849$. The resulting *mixed skew Weibull* model provides a better fit compared with the competition when applied to the entire data sets. Indeed, this can be seen from Table 3, which contains the results of fitting the same four models to the Australian currency, with each model being of the form (107) with $\hat{p}$ stated above.

Fit	**Australia**
Normal	0.069
AL	0.049
EP	0.028
*ADW**	**0.016**

Table 3: Kolmogorov-Smirnov distances between the data and four models calculated for entire data set (including zero returns) - Australian dollar. Note: reprinted from [43]

4 Univariate and Multivariate Weibull Distributions and their Applications - Review

The aim of this chapter is to review different existing variations of both, symmetric and asymmetric, univariate and multivariate Weibull distributions along with their basic mathematical properties and to put those generalizations into a broader context.
Over the years, people have been conducting numerous investigations in various fields connected with Weibull distributions in order to establish proper applications in the areas such as finance, insurance, economics, biostatistics, etc.
Recently, many researchers used Weibull distributions in modeling financial data as well as in survival analysis. Due to extensive use of the Weibull distribution in various areas, discovering the original sources of the information is almost impossible. Therefore, the purpose of this part of our work is to provide a concise resource of the most important properties of the univariate and multivariate Weibull distributions placed within one chapter. The facts are not necessary for our purposes but shed some light on various ways one can generalize the Weibull distribution.

4.1 Univariate Weibull distribution

Four different models are considered, each supported with pdf's, c.d.f.'s, moments or maximum likelihood estimation and application part if possible.
For the purpose of estimation, it is assumed that $X_1, X_2, ..., X_n$ is an i.i.d. sample and $X_{(1)} < X_{(2)} < .... < X_{(n)}$ are the corresponding order statistics.

4.1.1 Flaih- Elsalloukh-Mendi-Milanova's skewed double inverted Weibull distribution

Following the approach of Fernandez and Steel (see [19]), the authors introduced skewness into the symmetric inverted Weibull distribution obtaining the four-parameter Skewed Double Inverted Weibull distribution, named $SDIW_\alpha(\sigma, \kappa)$. The basic properties such as probability density function, cumulative distribution function, moments and maximum entropy are derived (see Flaih et al. [23]).

The symmetrization procedure (20) of the inverted Weibull distribution function

$$f(x) = \frac{\beta}{\alpha}\left(\frac{1}{x}\right)^{(\beta+1)} e^{-\frac{1}{\alpha}(\frac{1}{x})^\beta} \tag{108}$$

leads to a double inverted Weibull distribution function with density

$$f(x) = \frac{\beta}{2\alpha}|x|^{-(\beta+1)}e^{-\frac{1}{\alpha}}|x|^{-\beta}. \tag{109}$$

The skewness is introduced following the approach of Fernandez and Steel (see [19]), leading to the probability density function

$$f(x,\mu,\alpha,\beta,\epsilon) = \begin{cases} \frac{\beta}{2\alpha}\left(\frac{x-\mu}{1+\epsilon}\right)^{-(\beta+1)} e^{-\frac{1}{\alpha}\left(\frac{x-\mu}{1+\epsilon}\right)^{-\beta}}, & x \geq \mu \\ \frac{\beta}{2\alpha}\left(\frac{\mu-x}{1-\epsilon}\right)^{-(\beta+1)} e^{-\frac{1}{\alpha}\left(\frac{\mu-x}{1-\epsilon}\right)^{-\beta}}, & x < \mu, \end{cases} \tag{110}$$

where $\alpha > 0$ is a scale parameter, $\beta > 0$ is a shape parameter, μ is a location parameter, while $-1 < \epsilon < 1$ is the skewness parameter. The density is denoted by $SDIW_\beta(\alpha,\beta,\epsilon,\mu)$. The corresponding distribution function is:

$$F(x) = \begin{cases} 1 - \frac{1+\epsilon}{2}\left[1 - e^{-\frac{1}{\alpha}}\left(\frac{x-\mu}{1+\epsilon}\right)^{-\beta}\right], & x \geq \mu \\ \frac{1-\epsilon}{2}\left[1 - e^{-\frac{1}{\alpha}}\left(\frac{\mu-x}{1-\epsilon}\right)^{-\beta}\right], & x < \mu. \end{cases} \tag{111}$$

In the special case for $\beta = 1$, the density simplifies to:

$$f(x;\mu,\alpha,1,\epsilon) = \begin{cases} \frac{1}{2\alpha}\left(\frac{x-\mu}{1+\epsilon}\right)^{-2} e^{-\frac{1}{\alpha}\left(\frac{x-\mu}{1+\epsilon}\right)^{-1}}, & x \geq \mu \\ \frac{1}{2\alpha}\left(\frac{\mu-x}{1-\epsilon}\right)^{-2} e^{-\frac{1}{\alpha}\left(\frac{\mu-x}{1-\epsilon}\right)^{-1}}, & x < \mu, \end{cases} \tag{112}$$

which is the skewed double inverted exponential distribution, while the case $\beta = 2$ yields

$$f(x;\mu,\alpha,2,\epsilon) = \begin{cases} \frac{1}{\alpha}\left(\frac{x-\mu}{1+\epsilon}\right)^{-3} e^{-\frac{1}{\alpha}\left(\frac{x-\mu}{1+\epsilon}\right)^{-2}}, & x \geq \mu \\ \frac{1}{\alpha}\left(\frac{\mu-x}{1-\epsilon}\right)^{-3)} e^{-\frac{1}{\alpha}\left(\frac{\mu-x}{1-\epsilon}\right)^{-2}}, & x < \mu, \end{cases} \tag{113}$$

the skewed symmetric double inverted Rayleigh distribution.

When $1 \leq k \leq n$, the MLE's are found by the usual procedure of taking the derivatives of the log likelihood function. The MLE of α can be easily derived from the equation $\frac{\partial}{\partial\alpha}l_k(\beta,\alpha,\epsilon,\mu) = 0$, leading to

$$\hat{\alpha} = \left[\frac{1}{n}\sum_{i=k+1}^{n}\left(\frac{x_i-\mu}{1+\epsilon}\right)^{-\beta} + \sum_{i=1}^{k}\left(\frac{\mu-x_i}{1-\epsilon}\right)^{-\beta}\right], \tag{114}$$

while $\hat{\beta}_1$ and $\hat{\beta}_2$ can be calculated numerically by the use of some standard iterative procedure (i.e., Newton-Raphson method). Numerical methods must be applied to find other estimates as well.

The application part is illustrated by presenting two data sets. The first one, the number of million revolutions before failure for 23 endurance of deep - groove ball bearings, and the second one, the time intervals, in hours, between failures of the air conditioning system of an airplane. The MLE's of the unknown parameters and the log-likelihoods of four distributions are compared: $Gamma(\lambda, \alpha)$, $Weibull(\lambda, \alpha)$, $EE(\lambda, \alpha)$ and $SDIW(\beta, \alpha, \mu, \epsilon)$. While the first model did not show a good fit in terms of negative log-likelihood values, the second one proved to be the best outperforming other competitors. It can be concluded that $SDIW(\beta, \alpha, \mu, \epsilon)$ is very sensitive to the type of data, but at least in some cases works better than Gamma, Weibull or Exponential distribution.

4.1.2 Flaih-Elsalloukh-Mendi-Milanova's exponentiated inverted Weibull distribution

The authors proposed the extension of the standard inverted Weibull distribution (see Flaih et al. [22]) to the standard exponentiated inverted Weibull distribution by adding another shape parameter. The distribution function of standard exponentiated inverted Weibull distribution function (EIW) takes the form:

$$F_\theta(x) = (e^{-x^{-\beta}})^\theta, \qquad x, \beta, \theta > 0. \tag{115}$$

The corresponding density is:

$$f(x) = \theta\beta x^{-(\beta+1)}(e^{-x^{-\beta}})^\theta, \qquad x > 0. \tag{116}$$

For this distribution, the k-th moment is given by

$$E(X^k) = \int_0^\infty \theta\beta x^k x^{-(\beta+1)}(e^{-x^{-\beta}})^\theta dx = \theta^{\frac{k}{\beta}}\Gamma(1 - \frac{k}{\beta}), \qquad x > 0, \beta > k. \tag{117}$$

The maximum likelihood estimator of θ in terms of β is as follows:

$$\hat{\theta}(\beta) = \frac{n}{\sum_{i=1}^n x_i^{-\beta}}, \tag{118}$$

while the MLE of β is obtained as the fixed point solution of the non-linear equation of the form $h(\beta) = \beta$, where

$$h(\beta) = \beta - \frac{n}{\beta} + \sum_{i=1}^n \log x_i + \frac{n\sum_{i=1}^n x_i^{-\beta}\log x_i}{\sum_{i=1}^n x_i^{-\beta}} = 0. \tag{119}$$

To find $\hat{\beta}$ and $\hat{\theta}$ numerical methods are required. The least square estimators are obtained following the formulae:

$$\hat{\theta} = \exp\left(\frac{\sum_{i=1}^{n}\log^2 x_i \sum_{i=1}^{n}\log y_i + \sum_{i=1}^{n} y_i \log x_i \sum_{i=1}^{n}\log x_i}{(\sum_{i=1}^{n}\log x_i)^2 - n\sum_{i=1}^{n}\log^2 x_i}\right), \tag{120}$$

and

$$\hat{\beta} = \frac{n\sum_{i=1}^{n} y_i \log x_i - \sum_{i=1}^{n} y_i \sum_{i=1}^{n}\log x_i}{(\sum_{i=1}^{n}\log x_i)^2 - n\sum_{i=1}^{n}\log^2 x_i}. \tag{121}$$

The application part includes the following procedure. Two distributions, inverted Weibull distribution(IW) and exponentiated inverted Weibull distribution, (EIW) are fitted into the uncensored data set consisting of 100 observations concerning tensile strength of carbon fibers. Shape parameters θ and β were estimated along with the values of the log-likelihood functions. The log-likelihood ratio test has been performed provided a significantly better fit for the exponentiated inverted Weibull distribution EIW than inverted Weibull distribution.

4.1.3 Ali-Woo's skew-symmetric reflected distribution

In this paper the authors bring definitions of skew-symmetric distributions for a number of reflected distributions symmetric about zero (see [4]). For the purpose of this work, the skew reflected Weibull distribution is studied. As Balakrishnan and Kocherlakota (see [7]) previously described , the p.d.f. of the double Weibull distribution is obtained as

$$f(x) = \frac{\alpha}{2}|x|^{\alpha-1}e^{-|x|^{\alpha}}, x \in \mathbb{R}, \alpha > 0. \tag{122}$$

Using the standard signum function, denoted by sign(x), the above c.d.f can be expressed as:

$$F(x) = \frac{1}{2} + \frac{1}{2}\text{sign}(x)(1 - e^{-|x|^{\alpha}}). \tag{123}$$

Skewed Reflected (Double) Weibull Distribution proposed by Ali and Woo, denoted by $SDW_{\alpha}(1, c)$, is derived using Azzalini's approach of introducing skewness (see [3]) and the previous equation:

$$f(z; c) = \frac{\alpha}{2}|z|^{\alpha-1}e^{-|z|^{\alpha}}[1 + \text{sign}(cz)(1 - e^{-|cz|^{\alpha}})], z \in \mathbb{R}. \tag{124}$$

Adding a scale parameter σ we obtain a skewed reflected (double) Weibull Distribution denoted by $SDW_{\alpha}(\sigma, c)$:

$$f(z;c) = \frac{\alpha}{2\sigma}|\frac{z}{\sigma}|^{\alpha-1}e^{-|\frac{z}{\sigma}|^{\alpha}}[1+\text{sign}(\frac{cz}{\sigma})(1-e^{-|\frac{cz}{\sigma}|^{\alpha}})], z \in \mathbb{R}. \tag{125}$$

The corresponding c.d.f. arises from the previous equation and is given by:

$$F(z;c) = \begin{cases} 1-e^{-z^{\alpha}} + \frac{1}{2(1+c^{\alpha})}e^{-(1+c^{\alpha})z^{\alpha}}, z \geq 0, c > 0 \\ \frac{1}{2(1+c^{\alpha})}e^{-(1+c^{\alpha})(-z)^{\alpha}}, z < 0, c > 0. \end{cases} \tag{126}$$

For positive c, the k-th moment of the distribution is calculated as follows:

$$E(Z^k) = \Gamma\left(\frac{k}{\alpha}+1\right) + (-1+(-1)^k)\frac{1}{2}\frac{\Gamma(\frac{k}{\alpha}+1)}{(1+c^{\alpha})^{\frac{k}{\alpha}+1}}, \quad k = 1,2,3,.... \tag{127}$$

Next, the following procedure is related to the representation of the distribution showing the interesting connection between skew Laplace distribution and Ali-Woo model. If a random variable X follows a skew Laplace distribution with parameter $\lambda = c^{\alpha}$, $X \sim SL(\lambda = c^{\alpha})$, it can be proved that for $\lambda > 0$, $X^{<\frac{1}{\alpha}>} \overset{d}{=} SDW_{\alpha}(1,c)$. We start with the p.d.f. of the skew Laplace distribution which has the form:

$$f(x) = \frac{1}{2}e^{-|x|}(1+\text{sign}(\lambda x)(1-e^{-|\lambda x|})). \tag{128}$$

For $y < 0$, the c.d.f of $Y = X^{<\frac{1}{\alpha}>}$ is obtained as follows:

$$P(Y < y) = P(X < -|y|^{\alpha}) = P(X < -(-y)^{\alpha}) = F_x(-(-y)^{\alpha}) \tag{129}$$

and the corresponding p.d.f. can be easily found:

$$f(y) = F_x'(-(-y)^{\alpha})(-1)\alpha(-y)^{\alpha-1}(-1) = f_X(-(-y)^{\alpha})\alpha(-y)^{\alpha-1}. \tag{130}$$

Since f_X is a skew-Laplace, inserting into (128), the following expression is obtained:

$$g(y) = \frac{1}{2}\alpha|y|^{\alpha-1}e^{-|y|^{\alpha}}e^{-|\lambda||y|^{\alpha}}. \tag{131}$$

It can be easily verified that the above density is the Ali-Woo Skew Double Weibull model, $SDW_{\alpha}(1,\lambda)$, for $\lambda = c^{\alpha}$. For $y > 0$, by similar procedure, we obtain:

$$P(Y < y) = 1 - P(X > y^{\alpha}) = 1 - 1 - F_x(y^{\alpha}) = F_x(y^{\alpha}). \tag{132}$$

Taking the derivative the following expression arises:

$$f(y) = F'_X(y^\alpha)\alpha y^{\alpha-1} = f_X(y^\alpha)\alpha y^{\alpha-1}. \tag{133}$$

Inserting into (128), the Ali-Woo model with $\lambda = c^\alpha$ is obtained.

$$g(y) = \frac{1}{2}\alpha y^{\alpha-1} e^{-y^\alpha} e^{(2-|\lambda|y^\alpha)}. \tag{134}$$

4.1.4 Ali-Woo-Nadarajah's skew-symmetric (double) reflected inverted Weibull distribution

This model is obtained by applying the Azzalini's manner of introducing skewness into the Double Inverted Weibull Distribution $DIW_\alpha(\sigma, c)$ (see [5]). The model is presented with formulae for p.d.f and c.d.f., but no connection with the previous distributions seems to exist.
The definition of a skewed inverse reflected distribution is proposed by the authors having the p.d.f. specified by:

$$f(x) = \frac{\alpha}{2}\frac{1}{|x|^{\alpha+1}} e^{-\frac{1}{|x|^\alpha}} \left(1 + \text{sign}(cx) e^{-\frac{1}{|x|^\alpha c^\alpha}}\right).$$

Adding a scale parameter σ the formula for density is obtained:

$$f(x) = \frac{\alpha}{2\sigma}\frac{1}{|\frac{x}{\sigma}|^{\alpha+1}} e^{-\frac{1}{|\frac{x}{\sigma}|^\alpha}} \left(1 + \text{sign}(\frac{cx}{\sigma}) e^{-\frac{1}{|\frac{x}{\sigma}|^\alpha c^\alpha}}\right). \tag{135}$$

The corresponding c.d.f. takes the form:

$$F(x) = \begin{cases} \frac{1}{2}\left[1 - e^{-\frac{1}{(-\frac{x}{\sigma})^\alpha}} - \frac{c^\alpha}{c^\alpha+1} + \frac{c^\alpha}{c^\alpha+1} e^{-\frac{1}{(-\frac{x}{\sigma})^\alpha}\frac{c^\alpha+1}{c^\alpha}}\right], & x < 0 \\ \frac{1}{2}\left[1 + e^{-\frac{1}{(\frac{x}{\sigma})^\alpha}} - \frac{c^\alpha}{c^\alpha+1} + \frac{c^\alpha}{c^\alpha+1} e^{-\frac{1}{(\frac{x}{\sigma})^\alpha}\frac{c^\alpha+1}{c^\alpha}}\right], & x \geq 0. \end{cases} \tag{136}$$

Moments are given by:

$$E(X^n) = \frac{1}{2}\Gamma\left(1 - \frac{n}{\alpha}\right)\left[1 + (-1)^n + \left(\frac{c^\alpha}{1+c^\alpha}\right)^{1-\frac{n}{\alpha}}(1 - (-1)^n)\right] \tag{137}$$

$$E|X|^\eta = \Gamma\left(1 - \frac{n}{\alpha}\right). \tag{138}$$

4.2 Multivariate Weibul distributions

In this section the review of five different generalizations of exisiting multivariate Weibull distributions is presented. Most of the models are supported with formulae for p.d.f, c.d.f, MLE's and survival function.

4.2.1 Hanagal's multivariate Weibull distribution

The author presented a new multivariate Weibull distribution with many interesting properties. Knowing that there are many bivariate or multivariate Weibull distributions, (see [29]), based on bivariate or multivariate exponential distributions that are obtained as extensions of univariate exponential distribution, the reason Hanagal's multivariate model has been chosen is that this multivariate Weibull distribution (MVW) is obtained from multivariate exponential (MVE) model of Marshall-Olkin (see [69]) which is the MVE having the marginals as exponentials.
Based on this, if $\mathbf{Y} = (Y_1, Y_2, ..., Y_k)$ is $(k+1)$ parameter version of MVE model of Marshall-Olkin(see [69]) and Hanagal (see [29]), taking the transformations

$$X_i = Y_i^{\frac{1}{c}} \tag{139}$$

$c > 0$, $i = 1, ..., k$, the vector $\mathbf{X} = (X_1, X_2, ..., X_k)$, has, what one could call, multivariate Weibull distribution. Taking the survival function of Marshall-Olkin MVE model, (see [69])

$$\bar{F}_{\mathbf{Y}}(\mathbf{y}) = P[Y_1 > y_1, ..., Y_k > y_k] = \exp[-\sum_{i=1}^{k} \lambda_i y_i - \lambda_0 \max(y_1, ..., y_k)], \lambda_0, ..., \lambda_k > 0, \tag{140}$$

and the above transformation, the corresponding survival function (not absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^k$) of X of MVW is given by:

$$\bar{F}_{\mathbf{X}}(\mathbf{x}) = P[X_1 > x_1, ..., X_k > x_k] = \exp[-\sum_{i=1}^{k} \lambda_i x_i^c - \lambda_0 \max(x_1, ..., x_k)^c]. \tag{141}$$

As this distribution describes failure times and is derived from MVE of Marshall-Olkin, (see [69]) all applications of MVE will become real applications of this distribution as well (for e.g., simultaneous failure of nuclear power stations, simultaneous failure of hydroelectric pumps in aeroplanes, etc). The marginal distributions of $\mathbf{X} = (X_1, X_2, ..., X_k)$ are:

$$P[X_i > x_i] = \bar{F}_{\mathbf{X}}(0, ..., x_i, 0, ..., 0) = \exp[-(\lambda_i + \lambda_0)x_i^c], i = 1, ..., k, \tag{142}$$

which is the survival function of Weibull distribution with parameters $(\lambda_i + \lambda_0, c)$. Further, the MLEs of the parameters of MVW model are derived. The log-likelihood of the sample of size n is given by:

$$\begin{aligned}\log L = p\log c + n_0\log\lambda_0 + \sum_{i=1}^{k}\lambda_i\log\lambda_i \\ + \sum_{i=1}^{k} n_i(e)\log(\lambda_i+\lambda_0) + \sum_{i=1}^{k}(c-1)\sum_{j=1}^{n}\log x_{ij} \\ - \sum_{r=2}^{k}(r-1)(c-1)\sum_{j\in S_r}\log x_{(k)j} - \sum_{i=1}^{k}\lambda_i\sum_{j=1}^{n}x_{ij}^c - \lambda_0\sum_{j=1}^{n}x_{(k)j}^c,\end{aligned}$$

where $p = [nk - \sum_{r=2}^{k}(r-1)n_0(r)]$, $n_0(r)$ = number of observations with r of $X_i's$, $i = 1,...,k$ are equal, $n_0 = \sum_{r=2}^{k} n_0(r)$, n_i =number of observations in n which the random variable $X_i < X_{(k)}$,$n_i(e)$ = number of observations with X_i strictly the maximum of the $(X_1,...,X_k)$. The likelihood equations with respect to the parameters $(\lambda_0, \lambda_1, ..., \lambda_k, c)$ are:

$$n_0/\lambda_0 + \sum_{i=1}^{k} n_i(e)/(\lambda_i+\lambda_0) - \sum_{j=1}^{n} x_{(k)j}^c = 0 \tag{143}$$

$$n_i/\lambda_i + n_i(e)/(\lambda_i+\lambda_0) - \sum_{j=1}^{n} x_{ij}^c = 0, i = 1,...,k \tag{144}$$

$$\frac{p}{c} + \sum_{i=1}^{k}\sum_{j=1}^{n}\log x_{ij} - \sum_{r=2}^{k}(r-1)\sum_{j\in S_r}\log x_{(k)j} - \sum_{i=1}^{k}\lambda_i\sum_{j=1}^{n}x_{ij}^c\log x_{ij} - \lambda_0\sum_{j=1}^{n}x_{(k)}^c\log x_{(k)j} = 0. \tag{145}$$

The equations are difficult to solve algebraically, so the introduction of consistent estimators $(u_0, u_1, ..., u_{k+1})$ are needed as an initial solution in Newton-Raphson procedure or Fisher's method of scoring to obtain the MLE's $\hat{\lambda} = (\hat{\lambda}_0, \hat{\lambda}_1, ..., \hat{\lambda}_k, \hat{c})$.

4.2.2 Malevergne-Sornette's multivariate Weibull distributions

In this section a discussion of the non-Gaussian properties of the distributions of the asset returns is presented. The authors Malevergne and Sornette describe multivariate distributions for asset returns (see [66]) where marginal distributions are parameterized in terms of modified Weibull distribution. Two key parameters, c-the exponent and χ-the characteristic scale of the modified Weibull distribution are derived. Statistical tests of this parametrization are discussed.
Following previous work, the marginal probability distribution of returns is determined as follows :

$$f(x) = \frac{1}{2\pi} \frac{c}{\chi^{\frac{c}{2}}} |x|^{\frac{c}{2}-1} e^{-(\frac{|x|}{\chi})^c}, \tag{146}$$

where c and χ are the two key parameters. Taking into account a possible asymmetry between negative and positive returns (thus leading to possible non zero-average return), a more general parametrization takes form:

$$f(x) = \begin{cases} \frac{Q}{\pi} \frac{c_+}{\chi_+^{\frac{c_+}{2}}} |x|^{\frac{c_+}{2}-1} e^{-(\frac{|x|}{\chi_+})^c_+}, x \geq 0 \\ \frac{1-Q}{\pi} \frac{c_-}{\chi_-^{\frac{c_-}{2}}} |x|^{\frac{c_-}{2}-1} e^{-(\frac{|x|}{\chi_-})^c_-}, x < 0. \end{cases} \tag{147}$$

where Q (respectively $1-Q$) is the fraction of positive (respectively negative) returns. Only the case $Q = \frac{1}{2}$ will be considered since it is the only manageable case.

Transformation of the modified Weibull p.d.f. into a Gaussian law takes the form:

$$P(x_1, x_2, ..., x_N) = \frac{1}{2^N \pi^{\frac{N}{2}} \sqrt{\det V}} \left[-\sum_{i,j} V_{i,j}^{-1} \left(\frac{|x_i|}{\chi_i}\right)^{\frac{c}{2}} \left(\frac{|x_j|}{\chi_j}\right)^{\frac{c}{2}} \right] \left[\prod_{i=1}^{N} \frac{c_i |x_i|^{\frac{c}{2}-1}}{\chi_i^{\frac{c}{2}}} e^{-(\frac{|x|}{\chi})^c} \right] \tag{148}$$

where $V_i's$ are the components of the covariance matrix V.

4.2.3 Jye-Chyi's least square estimation for multivariate Weibull model of Hougaard based on accelerated life test

A simple method of incorporating the information collected from the accelerated life test on components and series system levels is introduced (see [44]). Due to Hogaard

(see [36]), the underlying distribution of life-times of components is assumed to be multivariate Weibull. The method of maximum likelihood estimation involves complex computational procedure and closed form solution is not obtained.
A system of m identical components whose lifetimes could be dependent is considered. Suppose that life-times of m components, denoted as $Z_1, ..., Z_m$, are identically distributed. Regardless whether the components are assembled into a system, the Hougaard multivariate Weibull distribution MVW (see [36]) of modeling components life-times, is applied. The corresponding survival function of the MVW takes form ,

$$\bar{F}_{\mathbf{Y}}(z_1, z_2, ..., z_m) = \exp\left\{ -\left[\sum_{k=1}^{m}\left(\frac{z_k}{\theta_k}\right)^{\frac{\beta_k}{\delta}}\right]^{\delta}\right\}, \quad \delta \in (0,1], \beta_i, \theta_i > 0, i = 1, ..., k, \quad z_1, ..., z_m \geq 0. \tag{149}$$

This distribution includes important properties such as a physical motivation (see [36]), existence of absolutely continuous probability density function, Weibull marginals and stability relations formulated in Tawn (see [91]).

4.2.4 Crowder's multivariate distribution with Weibull connections

In this section we describe a simple form of multivariate distribution which, for certain parameter values, has Weibull marginals (see [13]). A single parameter taking positive, negative or zero values is included in the distribution. The marginals and conditional distributions can be presented in a simple forml which makes the distribution suitable for practical application and interpretation. It is known that a parametric analysis of failure time data, both in univariate and multivariate case, is widely modeled using the Weibull distribution. Some discussions are made regarding the distribution of survival times really being Weibull (see [80], [81]). Johnson and Kotz (see [37]) propose multivariate exponential distribution which leads to Weibull distribution by taking power transformations of each variable.

Suppose $T_1, T_2, , ...T_p$ are independent Weibull random variables with survival function

$$P(T_j > t_j) = \exp(-\alpha_j t_j^{\phi_j}) \tag{150}$$

and log-means $E[\ln T_j] = -\Phi_j^{-1}(\ln \alpha_j + \gamma)$, where γ is the Euler's constant. Taking into account the normal linear mixed-effects model, the author suggests (see [12]) taking $\ln \alpha_j = \ln \xi + \ln \alpha$, where ξ is a fixed effects parameter and α varies randomly over individuals.

The same paper assumed that a multivariate Burr distribution for $T = (T_1, T_2, , ..., T_p)$ resulted from the assumption that α has a Gamma distribution, while in view of the central limit theorem and corresponding normal random effects model, $\ln \alpha$ is normally distributed.

If we assume that α has a stable distribution with distribution function G on $(0, \infty)$ of characteristic exponent ν, (see Feller [18]), T has joint survival function

$$P(T > t) = \int_0^\infty \exp\left(-\sum_{j=1}^{p} \alpha_j t_j^{\phi_j}\right) dG_\nu(\alpha) = \exp(-s^\nu), \tag{151}$$

where $s = \sum \xi_j t_j^{\phi_j}$ which is a multivariate distribution having Weibull marginals.

4.2.5 Hsiaw-Chan-Yeh's multivariate semi-Weibull distribution

The author (see [35]) proposes two more general multivariate distributions with Weibull marginals that are constructed following Lee (see [64]) and Marshall and Olkin (see [70]) results. The first one, named the Marshall-Olkin multivariate semi-Weibull distribution is denoted as $MO - MSW$. The second one is termed the multivariate semi Weibull distribution (and denoted as MSW). Here, more general cases are considered.

A distribution with parameters $p \in (0, 1)$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k) \geq \mathbf{0}$ with survival function of the form

$$\bar{F}_{\mathbf{x}}(\mathbf{X}) = e^{-\Psi(\mathbf{x})} \tag{152}$$

is said to be a k-variate semi-Weibull distribution if $\Psi(\mathbf{X})$ satisfies the functional equation

$$\Psi(\mathbf{x}) = \frac{1}{p}\Psi(p^{\frac{1}{\alpha_1}} x_1, p^{\frac{1}{\alpha_2}} x_2, ..., p^{\frac{1}{\alpha_k}} x_k), \quad \mathbf{x} \geq 0. \tag{153}$$

$\Psi(\mathbf{x})$ is nonnegative and monotonically increasing in all x_i. This MSW is considered homogeneous if all $\alpha_i's$ are equal. If $\bar{F}_{\mathbf{X}}$ is the joint survival function of the k-variate random vector $\mathbf{X}$, then the function

$$\bar{G}(\mathbf{x}) = \frac{\beta \bar{F}_{\mathbf{X}}}{1 - (1-\beta)\bar{F}(\mathbf{x})}, \quad x \geq 0, \quad 0 < \beta \leq 1, \tag{154}$$

is a proper k-variate survival function. The family of distributions of this form is defined as Marshall-Olkin multivariate family of distributions.

The Marshal-Olkin multivariate Weibull distribution is of the form:

$$\bar{F}_{\mathbf{X}}(\mathbf{x}) = \exp\left\{-\sum_{i=1}^{k}\lambda_i x_i^{\alpha_i} + \sum_{1=i}^{k-1}\sum_{j=2}^{k}\lambda_{ij}\max(x_i^{\alpha_i}, x_j^{\alpha_j}) + ... + \lambda_{12...k}\max(x_1^{\alpha_1}...x_k^{\alpha_k})\right\} \tag{155}$$

with all $\lambda_i, \lambda_{ij}, \lambda_{12...k} > 0$ and $\alpha = (\alpha_1, ..., \alpha_k) > \mathbf{0}$ and $\mathbf{X} > \mathbf{0}$. The function

$$\Psi(\mathbf{x}) = \sum_{i=1}^{k}\lambda_i x_i^{\alpha_i} + \sum_{1=i}^{k-1}\sum_{j=2}^{k}\lambda_{ij}\max(x_i^{\alpha_i}, x_j^{\alpha_j}) + ... + \lambda_{12...k}\max(x_1^{\alpha_1}...x_k^{\alpha_k}) \tag{156}$$

satisfies the functional equation of the survival function, so it can be concluded that it belongs to the MSW distributions. For any $p \in (0, 1)$ we have

$$\begin{aligned}\Psi(\mathbf{x}) = &\sum_{i=1}^{k}\lambda_i(p^{1/\alpha_i}x_i)^{\alpha_i} + \\ &+ \sum_{1=i}^{k-1}\sum_{j=2}^{k}\lambda_{ij}\max((p^{1/\alpha_i}x_i)^{\alpha_i}, (p^{1/\alpha_j}x_j)^{\alpha_j}) + ... + \lambda_{12...k}\max_{1\le i\le k}((p^{1/\alpha_i}x_i)^{\alpha_i}) \\ &= \Psi(p^{1/\alpha_1}x_1, p^{1/\alpha_2}x_2, ..., p^{1/\alpha_k}x_k) = \Psi(p^{1/\alpha}\mathbf{x}).\end{aligned} \tag{157}$$

The same can be proved for multivariate extension of Lee's survival functions (see [64]):

$$\bar{F}_{\mathbf{X}}(\mathbf{x}) = e^{-(\sum_{i=1}^{k}x_i^4)^{\frac{1}{2}}} \tag{158}$$

and

$$\bar{F}_{\mathbf{X}}(\mathbf{x}) = e^{-(\sum_{i=1}^{k}x_i^4)^{1/2}}e^{-\sum_{i=1}^{k}\lambda_i^2 x_i^2 + \sum_{1=i}^{k-1}\sum_{<j=2}^{k}\lambda_{ij}\max(x_i^2, x_j^2) + ... + \lambda_{12...k}\max(x_1^2...x_k^2)}. \tag{159}$$

5 Asymmetric multivariate Weibull distribution

5.1 Definition and basic properties

A class of multivariate laws as an extension of univariate Weibull distribution is presented in this chapter. The well known representation of the asymmetric univariate Laplace distribution is used as the starting point. Properties of this new family possess some similarities to the multivariate normal distribution and are explored including moments, correlations, densities and simulation algorithms.

As was said before, skewness has been introduced into the symmetric double Weibull distribution by Fernandez and Steel (see [19]). Two inverse scale factors transform a symmetric distribution with p.d.f. f into an asymmetric one with the p.d.f.

$$g(x) = \frac{\kappa}{\sigma(1+\kappa^2)} \begin{cases} f(x\kappa/\sigma), & x > 0 \\ f(\frac{x}{\sigma\kappa}), & x < 0, \end{cases} \tag{160}$$

where $\kappa > 0$. This distribution will be called the asymmetric Weibull distribution with parameters κ and σ. However, this procedure does not lend itself to generalization to multivariate distributions.

To this end recall that the standard Laplace random variable Y admits the representation

$$Y \stackrel{d}{=} \sqrt{2E}Z, \tag{161}$$

where E is standard exponential and Z is standard normal independent of E, (see Kotz et al. [49]). This representation shows that the distribution of Y is a scale mixture of normal distributions. Furthermore, Kozubowski and Podgórski (see [48]) show that the random variable

$$Y \stackrel{d}{=} mE + \sqrt{2E}Z \tag{162}$$

has the asymmetric Laplace distribution in the sense of (160). On the other hand, let L have the symmetric Laplace distribution and let S be an independent stable random variable with index $\alpha \in (0, 1]$ defined by the Laplace transform

$$g(t) = \mathbb{E}e^{-tS} = \int_0^\infty e^{-st} f_S(s) ds = e^{-t^\alpha}. \tag{163}$$

It can be shown by an elementary calculation that the random variable $Y = L/S$ has the symmetric Weibull distribution with parameters α and $\sigma = 1$. This together with (162) leads to the idea that the random variable W defined by

$$W \stackrel{d}{=} \frac{mE + \sqrt{2E}X}{S} \tag{164}$$

with E, $X \sim \mathrm{N}(0, \tau^2)$ and S independent has the asymmetric Weibull distribution.

Theorem 5.1 *The random variable* $W \stackrel{d}{=} \frac{mE+\sqrt{2E}X}{S}$, *where* E *is a standard exponential random variable,* $E \sim exp(1)$, S *standard stable random variable independent of* E *given by the Laplace transform (8) and* X *normal random variable,* $X \sim \mathrm{N}(0, \tau^2)$ *has the* $\mathcal{ADW}_\alpha(\sigma, \kappa)$ *distribution with parameters*

$$\sigma = \tau \qquad \text{and} \qquad \kappa = \frac{\sqrt{m^2 + 4\tau^2} - m}{2\tau}.$$

Proof: Since $W = \frac{mE+\sqrt{2E}X}{S}$ we define a function $\Phi : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^3$ as

$$\Phi(u, s, x) = (u, s, \frac{mu + \sqrt{2u}x}{s}). \tag{165}$$

Then

$$\Phi^{-1}(u, s, w) = (u, s, \frac{ws - mu}{\sqrt{2u}}). \tag{166}$$

Knowing that Jacobian J of the function Φ^{-1} is

$$J_{\Phi^{-1}}(u, s, x) = \frac{s}{\sqrt{2u}}, \tag{167}$$

it follows that p.d.f. of r.v. (E, S, W) is:

$$f_{E,S,W}(u, s, w) = e^{-u} f_S(s) \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(ws-mu)^2}{4\sigma^2 u}} \frac{s}{\sqrt{2u}}. \tag{168}$$

Elementary calculation yields:

$$f_{E,S,W}(u, s, w) = e^{-u} f_S(s) \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{mws}{2\sigma^2}} e^{-\frac{w^2 s^2}{4u\sigma^2}} e^{-\frac{m^2 u}{4\sigma^2}} \frac{s}{\sqrt{2u}}. \tag{169}$$

Integrating over u and applying formula 5.28 Oberhettinger and Badii

$$\int_0^\infty \frac{1}{\sqrt{t}} e^{-\frac{a}{t} - pt} dt = \left(\frac{\pi}{p}\right)^{\frac{1}{2}} e^{-2\sqrt{ap}} \tag{170}$$

(see [79]), we get:

$$f_{S,W}(s, w) = \frac{s}{2\sigma\sqrt{1 + \frac{m^2}{4\sigma^2}}} e^{\frac{mws}{2\sigma^2}} e^{-2\sqrt{1+\frac{m^2}{4\sigma^2}}\sqrt{\frac{w^2 s^2}{4\sigma^2}}} f_S(s). \tag{171}$$

After some simplification we obtain:

$$f_{S,W}(s, w) = \frac{s}{2\sqrt{4\sigma^2 + m^2}} e^{-\frac{s}{2\sigma^2}[\sqrt{4\sigma^2+m^2}|w| - mw]} f_S(s). \tag{172}$$

Case 1: $w > 0$

$$f_{S,W}(s,w) = \frac{s}{2\sqrt{4\sigma^2+m^2}} e^{-\frac{s}{2\sigma^2}[(\sqrt{4\sigma^2+m^2}-m)w]} f_S(s). \tag{173}$$

Since $E(Se^{-\lambda S}) = \alpha\lambda^{\alpha-1}e^{-\lambda^\alpha}$ it follows:

$$f_W(w) = \frac{\alpha}{2\sqrt{4\sigma^2+m^2}} \left[\frac{(\sqrt{4\sigma^2+m^2}-m)w}{2\sigma^2}\right]^{\alpha-1} e^{-(\frac{(\sqrt{4\sigma^2+m^2}-m)w}{2\sigma 2})^\alpha} \tag{174}$$

Case 2: $w < 0$

$$f_{S,W}(s,w) = \frac{s}{2\sqrt{4\sigma^2+m^2}} e^{-\frac{s}{2\sigma^2}[(\sqrt{4\sigma^2+m^2}+m)(-w)]} f_S(s), \tag{175}$$

which gives

$$f_W(w) = \frac{\alpha}{2\sqrt{4\sigma^2+m^2}} \left[\frac{(\sqrt{4\sigma^2+m^2}+m)(-w)}{2\sigma^2}\right]^{\alpha-1} e^{-(\frac{-w(\sqrt{4\sigma^2+m^2}+m)}{2\sigma 2})^\alpha}. \tag{176}$$

It follows that f_W has appropriate form with:

$$\frac{\kappa}{\tau} = \frac{\sqrt{4\sigma^2+m^2}-m}{2\sigma^2} \tag{177}$$

and

$$\frac{1}{\kappa\tau} = \frac{\sqrt{4\sigma^2+m^2}+m}{2\sigma^2}. \tag{178}$$

By multiplying the equations we obtain

$$\sigma = \tau \quad \text{and} \quad \kappa = \frac{\sqrt{m^2+4\sigma^2}-m}{2\sigma}. \tag{179}$$

Next, we want to check the constants in the formula for density of $\mathcal{ADW}_\alpha$ by calculating cases for $w > 0$ and $w < 0$.
First,

$$\frac{1}{\kappa} = \frac{2\sigma}{\sqrt{m^2+4\sigma^2}-m}, \tag{180}$$

which, upon multiplying by $\sqrt{m^2+4\sigma^2}+m$, leads to:

$$\kappa + \frac{1}{\kappa} = \frac{\sqrt{m^2+4\sigma^2}}{\sigma} \quad \text{and} \quad \frac{\kappa}{\kappa^2+1} = \frac{\sigma}{\sqrt{m^2+4\sigma^2}}. \tag{181}$$

Case 1: For $w > 0$ we obtain

$$\frac{1}{\sigma^\alpha}\frac{\alpha\kappa}{1+\kappa^2}\kappa^{\alpha-1} = \frac{1}{\sigma^\alpha}\frac{\alpha\sigma}{\sqrt{m^2+4\sigma^2}}\left(\frac{\sqrt{m^2+4\sigma^2}-m}{2\sigma}\right)^{\alpha-1}, \tag{182}$$

leading to

$$\frac{\alpha}{\sqrt{m^2+4\sigma^2}}\left(\frac{\sqrt{m^2+4\sigma^2}-m}{2\sigma^2}\right)^{\alpha-1}. \tag{183}$$

Case 2: Similarly, for $w < 0$ we have

$$\frac{1}{\sigma^\alpha}\frac{\alpha\kappa}{1+\kappa^2}\frac{1}{\kappa^{\alpha-1}} = \frac{1}{\sigma^\alpha}\frac{\alpha\sigma}{\sqrt{m^2+4\sigma^2}}\frac{1}{(\frac{\sqrt{m^2+4\sigma^2}-m}{2\sigma})^{\alpha-1}}. \tag{184}$$

Upon multiplying the latter fraction by $(\sqrt{4\sigma^2+m^2}+m)^{\alpha-1}$, the following expression is obtained:

$$\frac{\alpha}{\sqrt{m^2+4\sigma^2}}\left(\frac{\sqrt{m^2+4\sigma^2}+m}{2\sigma^2}\right)^{\alpha-1}. \tag{185}$$

The representation (164) leads to a multidimensional generalization of the univariate asymmetric Weibull distribution. We define

$$\mathbf{W} = \frac{\mathbf{m}E+\sqrt{2E}\mathbf{X}}{S} \tag{186}$$

where $\mathbf{m} \in \mathbb{R}^d$, and $\mathbf{\Sigma}$ is a $d \times d$ positive semi definite symmetric matrix, and the notation $\mathbf{X} \sim \mathrm{N}_d(\mathbf{0}, \mathbf{\Sigma})$ is used to indicate a d-dimensional normal distribution with the mean vector $\mathbf{0}$ and the covariance matrix $\mathbf{\Sigma}$. Note that by (12) the marginal distributions of $\mathbf{W}$ are asymmetric Weibull. This justifies the name asymmetric multivariate Weibull distribution.

Symmetric and asymmetric versions of various distributions have been used to model asset returns and currency exchange rates, (see [43], [52], [66], [74] and [83]). The above generalization provides a new family of distributions which can potentially be used in modeling multivariate financial data. The components will never be independent but the advantage of this generalization is in the fact that this family of distributions inherits nice properties of the multivariate normal distribution. Linear combinations of components are asymmetric Weibull. The other limitation is that $\alpha \in (0, 1]$ but it can be shown that only for such α we get a unimodal distribution which is of advantage for modeling purposes. The chapter continues with the representation followed by distributions of linear combinations, conditional distributions, moments and the covariance matrix.

Definition 5.1 *A random vector* $\mathbf{W} \in \mathbb{R}^d$ *has a multivariate Weibull distribution with parameters* $0 < \alpha \leq 1$, $\mathbf{m} \in \mathbb{R}^d$, *and* $\mathbf{\Sigma}$ *is a* $d \times d$ *positive semi definite symmetric matrix, denoted by* $\mathrm{W}_d(\alpha, \mathbf{m}, \mathbf{\Sigma})$ *if the following representation holds:*

$$\mathbf{W} = \frac{\mathbf{m}E + \sqrt{2E}\mathbf{X}}{S}, \tag{187}$$

where E *is a standard exponential random variable,* S *standard stable random variable independent of* E *given by the Laplace transform (8), and* $\mathbf{X} \in \mathbb{R}^d$ *a multivariate normal random vector centered at zero with the covariance matrix* $\mathbf{\Sigma}$*, denoted by* $\mathbf{X} \sim \mathrm{N}_d(0, \mathbf{\Sigma})$ *independent of* (E, S).

For $\mathbf{m} = 0$, the symmetric case is obtained while for $\alpha = 1$ the variable S is constant and equal to 1 so that W has an asymmetric Laplace distribution (with Laplace marginals). For $\alpha = \frac{1}{2}$, the variable S has the Lévy density

$$f_S(s) = \frac{1}{2\sqrt{\pi s^3}} e^{-\frac{1}{4s}} \ , s > 0. \tag{188}$$

5.2 Polar representation

The class of elliptically symmetric distributions consists of laws with non singular $\mathbf{\Sigma}$ and the density

$$f(\mathbf{x}) = k_d |\mathbf{\Sigma}|^{-\frac{1}{2}} g[(\mathbf{x} - \mathbf{m})' \mathbf{\Sigma}^{-1} (\mathbf{x} - \mathbf{m})], \tag{189}$$

where g is a one-dimensional real valued function independent of d, and k_d is a proportionality constant. We will denote the laws with the density (189) by $EC_d(\mathbf{m}, \mathbf{\Sigma}, g)$. Every r.v. $\mathbf{Y} \sim EC_d(\mathbf{0}, \mathbf{\Sigma}, g)$ admits the representation (see Fang et al. [17])

$$\mathbf{Y} \stackrel{d}{=} R\mathbf{H}\mathbf{U}^d, \tag{190}$$

where $\mathbf{H}$ is a $d \times d$ matrix such that $\mathbf{H}\mathbf{H}' = \mathbf{\Sigma}$, R is a positive r.v. independent of $\mathbf{U}^d$ (having the distribution of $\sqrt{\mathbf{Y}'\mathbf{\Sigma}^{-1}\mathbf{Y}}$), and $\mathbf{U}^d$ is a r.v. uniformly distributed on the sphere S_{d-1} , so that $\mathbf{H}\mathbf{U}^d$ is uniformly distributed on the surface of the hyperellipsoid $\left\{\mathbf{y} \in \mathbb{R}^d : \mathbf{y}\mathbf{\Sigma}^{-1}\mathbf{y} = 1\right\}$, (see Kotz et al. [48]).

Proposition 5.1 *Let* $\mathbf{Y} \sim W_d(\alpha, 0, \mathbf{\Sigma})$ *where* $\mathbf{\Sigma} > 0$.*Then,* $\mathbf{Y}$ *admits the polar representation(190), where* $\mathbf{H}$ *is* $d \times d$ *matrix such that* $\mathbf{H}\mathbf{H}' = \mathbf{\Sigma}$, $\mathbf{U}^d$ *is a r.v. uniformly distributed on the sphere* S_{d-1}*, and* R *is a positive r.v. independent of* $\mathbf{U}^d$ *with the density*

$$f_R(z) = \frac{\sqrt{2}d}{2^{\frac{d}{2}}\Gamma\left(\frac{d}{2}+1\right)} \int_0^\infty \left(\frac{z}{y}\right)^{d-1} e^{-\frac{1}{2}\left(\frac{z}{y}\right)^2} \int_0^\infty \sqrt{x} e^{-y^2 x} f_S(\sqrt{x}) dx dy. \tag{191}$$

Proof: By Definition 5.1 the vector $\mathbf{Y}$ has the representation (187). Take $\mathbf{m} = 0$ and $\mathbf{\Sigma} = \mathbf{HH}'$, where $\mathbf{H}$ is a $d \times d$ non-singular lower triangular matrix. The r.v. $\mathbf{X} \sim \mathrm{N}_d(0, \mathbf{\Sigma})$ has the representation $\mathbf{X} = \mathbf{HN}$, where $\mathbf{N} \sim \mathrm{N}_d(0, \mathbf{I})$. Further, the r.v. $\mathbf{N}$ allows the representation $\mathbf{N} \overset{d}{=} R_N \mathbf{U}^d$, where R_N and $\mathbf{U}^d$ are independent and $\mathbf{U}^d$ is uniformly distributed on S_d while $R_N \sim \chi^2(d)$. Therefore, it is sufficient to show that $\sqrt{\frac{2E}{S^2}} R_N$ has density (191) which follows by an elementary calculation.

Note that for invertible $\mathbf{\Sigma}$, the multivariate Weibull distribution has a density. By independence, we have that

$$(\mathbf{W}|E = u, S = s) \sim \mathrm{N}_d\left(\frac{\mathbf{m}u}{s}, \frac{2u}{s^2}\mathbf{\Sigma}\right) . \tag{192}$$

The unconditional density will be computed in Chapter 6 for the case $d = 2$.

5.3 Linear transformations

The multivariate asymmetric Weibull distribution inherits some of the properties of the multivariate normal distribution.

Proposition 5.2 *Let* $\mathbf{W} = (W_1, W_2, ..., W_d)' \sim W_d(\alpha, \mathbf{m}, \mathbf{\Sigma})$ *and let* $\mathbf{A}$ *be an* $l \times d$ *real matrix. Then, the random vector* $\mathbf{AW}$ *is* $W_l(\alpha, \mathbf{m}_\mathbf{A}, \mathbf{\Sigma}_\mathbf{A})$ *where*

$\mathbf{m}_\mathbf{A} = \mathbf{Am}$ *and* $\mathbf{\Sigma}_\mathbf{A} = \mathbf{A\Sigma A}'$.

Proof: We note

$$\mathbf{AW} = \frac{\mathbf{A}(\mathbf{m}E + \sqrt{2E}\mathbf{X})}{S} = \frac{\mathbf{Am}E + \sqrt{2E}\mathbf{AX}}{S} \tag{193}$$

where the random vector $\mathbf{AW}$ is centered at $\mathbf{m}_\mathbf{A} = \mathbf{Am}$ and $\mathbf{AX} \sim \mathrm{N}_l(\mathbf{0}, \mathbf{A\Sigma A}')$. This concludes the proof.
In particular, it follows that all univariate and multivariate marginals as well as linear combinations of the components of a multivariate Weibull are multivariate Weibull.

Corollary 5.1 *Let* $\mathbf{W} = (W_1, W_2, ..., W_d)' \sim W_d(\alpha, \mathbf{m}, \mathbf{\Sigma})$, *where* $\mathbf{\Sigma} = (\sigma_{i,j})_{i,j=1}^d$. *Then,*
(i) For all $k \le d$, $(W_1, ..., W_k) \sim W_k(\alpha, \mathbf{m}_\mathbf{k}, \mathbf{\Sigma}_\mathbf{k})$, *where* $\mathbf{m}_\mathbf{k} = (m_1, ..., m_k)$ *and* $\mathbf{\Sigma}_\mathbf{k}$ *is a* $k \times k$ *matrix with* $\sigma'_{ij} = \sigma_{ij}$ *for* $i, j = 1, ...k$;
(ii) For any $\mathbf{b} = (b_1, ..., b_d)' \in \mathbf{R}^d$, *the r.v.* $W_\mathbf{b} = \sum_{k=1}^d b_k W_k$, *is univariate asymmetric Weibull random variable with* $\sigma = \sqrt{\mathbf{b}'\mathbf{\Sigma b}}$ *and* $\mu = \mathbf{m}'\mathbf{b}$.
(iii) Marginal distributions are again multivariate asymmetric Weibull.

5.4 Conditional distributions

Consider $\mathbf{W} \sim W_d(\alpha, \mathbf{m}, \mathbf{\Sigma})$. It is of some interest to consider conditional distributions. Let

$$\mathbf{W} = \begin{pmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \end{pmatrix} \sim W_d(\alpha, \mathbf{m}, \mathbf{\Sigma})$$

with corresponding dimensions d_1 and d_2, $d_1 + d_2 = d$ and

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \sim \mathrm{N}_d(\mathbf{0}, \mathbf{\Sigma}),$$

with

$$\mathbf{m} = \begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix} \qquad \text{and} \quad \mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{pmatrix},$$

where Σ_{ij} are elements of the matrix $\mathbf{\Sigma}$. With the above notation we have:

Proposition 5.3 *Assume that $|\mathbf{\Sigma}_{22}| > 0$ and $\mathbf{m}_2 = \mathbf{0}$. The distribution of*

$$\mathbf{W}_1 - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{W}_2 \tag{194}$$

is $W_{d_1}\left(\alpha, \mathbf{m}_1, \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}\right)$.

Proof: The vector $\mathbf{X}_1 - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2$ is independent of $(E, S, \mathbf{W}_2)$. This means that

$$\frac{\mathbf{m}_1 E + \sqrt{2E}\left(\mathbf{X}_1 - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2\right)}{S} \sim W_{d_1}\left(\alpha, \mathbf{m}_1, \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}\right).$$

Note that by independence

$$\left(\mathbf{X}_1 - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2 | E = u, S = s, \mathbf{W}_2 = \mathbf{w_2}\right) \sim \mathrm{N}_{d_1}\left(\mathbf{0}, \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}\right)$$

where

$$\mathbf{w}_2 = \frac{\mathbf{m}_2 u + \sqrt{2u}\mathbf{x}_2}{s}.$$

5.5 Expectations and covariances

First we need negative moments of S. We note that on the one hand

$$\int_0^\infty t^{\beta-1}\mathbb{E}(e^{-tS})dt = \int_0^\infty t^{\beta-1}e^{-t^\alpha}dt = \frac{1}{\alpha}\Gamma\left(\frac{\beta}{\alpha}\right) \tag{195}$$

and on the other

$$\int_0^\infty t^{\beta-1}\mathbb{E}(e^{-tS})dt = \mathbb{E}\left(\int_0^\infty t^{\beta-1}e^{-tS}dt\right) = \mathbb{E}\left(\frac{\Gamma(\beta)}{S^\beta}\right). \tag{196}$$

It follows that

$$\Gamma(\beta)\mathbb{E}[S^{-\beta}] = \frac{1}{\alpha}\Gamma\left(\frac{\beta}{\alpha}\right) = \frac{1}{\beta}\Gamma\left(\frac{\beta}{\alpha}+1\right). \tag{197}$$

Taking $\beta = n$ we get

$$\mathbb{E}[S^{-n}] = \frac{1}{n}\frac{\Gamma(\frac{n}{\alpha}+1)}{\Gamma(n)} = \frac{\Gamma\left(\frac{n}{\alpha}+1\right)}{\Gamma(n+1)} \tag{198}$$

Let $\mathbf{W} \sim W_d(\alpha, \mathbf{m}, \mathbf{\Sigma})$. By independence of E, S and $\mathbf{X}$ in the Definition 5.1, it follows that

$$\mathbb{E}(W_i|E,S) = \frac{m_i E}{S} \quad \text{and} \quad \operatorname{cov}(W_i, W_j|E,S) = \frac{2E}{S^2}\sigma_{ij}\,. \tag{199}$$

We compute

$$\mathbb{E}\left(\mathbf{W}\right) = \mathbb{E}\left(\mathbb{E}\left(\mathbf{W}|E,S\right)\right) = \mathbf{m}\Gamma\left(\frac{1}{\alpha}+1\right) \tag{200}$$

and

$$\begin{aligned} \operatorname{cov}(W_i, W_j) &= \mathbb{E}\left(\operatorname{cov}\left(W_i, W_j|E,S\right)\right) + \operatorname{cov}\left(\mathbb{E}\left(W_i|E,S\right), \mathbb{E}\left(W_j|E,S\right)\right) \\ &= \sigma_{ij}\mathbb{E}\left(\frac{2E}{S^2}\right) + \operatorname{cov}\left(\frac{m_i E}{S}, \frac{m_j E}{S}\right) \\ &= \sigma_{ij}\Gamma\left(\frac{2}{\alpha}+1\right) + m_i m_j\left[\Gamma\left(\frac{2}{\alpha}+1\right) - \Gamma\left(\frac{1}{\alpha}+1\right)^2\right]. \end{aligned} \tag{201}$$

6 Densities and simulation

Applications of the $W_d(\alpha, \mathbf{m}, \mathbf{\Sigma})$ depend on effective estimation methods. To apply maximum likelihood explicit densities are needed. However, it is known that the densities of the standard stable random variable are known explicitly in special cases like $\alpha = 1/2$ only. Assume that $\mathbf{\Sigma}$ is invertible. From Definition 5.1 and linear transformation properties of multivariate normal distribution, it follows that

$$f_{\mathbf{W}|E=u,S=s}(\mathbf{w}) = \frac{s^n}{(2\pi)^{d/2}(2u)^{d/2}\sqrt{|\mathbf{\Sigma}|}} \exp\left[-\frac{s^2}{4u}\left(\mathbf{w} - \frac{\mathbf{m}u}{s}\right)^T \mathbf{\Sigma}^{-1}\left(\mathbf{w} - \frac{\mathbf{m}u}{s}\right)\right]. \tag{202}$$

Denote

$$a = \mathbf{w}'\mathbf{\Sigma}^{-1}\mathbf{w}, \quad b = \mathbf{w}'\mathbf{\Sigma}^{-1}\mathbf{m} \quad \text{and} \quad c = \mathbf{m}'\mathbf{\Sigma}^{-1}\mathbf{m}. \tag{203}$$

Rewriting (202) we get

$$f_{\mathbf{W}|E=u,S=s}(\mathbf{w}) = \frac{s^n}{(2\pi)^{d/2}(2u)^{d/2}\sqrt{|\mathbf{\Sigma}|}} \exp\left[-\frac{as^2}{4u} + \frac{bs}{2} - \frac{cu}{4}\right]. \tag{204}$$

We multiply by $f_E(u)f_S(s)$ and first integrate over u to obtain the joint density of $\mathbf{W}$ and S. By formula (5.34) in [79] we have for $\mu, p, q > 0$

$$\int_0^\infty \frac{1}{u^\mu} e^{-\frac{p}{t} - qt} dt = 2\left(\frac{p}{q}\right)^{-\mu/2+1/2} K_{\mu-1}\left(2\sqrt{pq}\right) \tag{205}$$

where $K_\nu(z)$ is the modified Bessel function of index ν. Let $\beta = \sqrt{a\left(\frac{c}{4}+1\right)}$. Using (297) we get after some computation that

$$f_{\mathbf{W},S}(\mathbf{w}, s) = \frac{s^{d/2+1}}{(2\pi)^{d/2}\sqrt{|\mathbf{\Sigma}|}} e^{\frac{bs}{2}} \left(\frac{\beta}{a}\right)^{d/2-1} K_{d/2-1}(\beta s) f_S(s). \tag{206}$$

By formula (3.20) in [79] the function $K_\nu(p)$ has the representation

$$K_\nu(p) = \frac{\sqrt{\pi}(p/2)^\nu e^{-p}}{\Gamma(\nu + 1/2)} \int_0^\infty e^{-pt}\left[t(t+2)\right]^{\nu-1/2} dt. \tag{207}$$

for $p > 0, \nu > -1/2$. Denote for $y > 0$

$$\phi_d(y) = E\left(S^d e^{-yS}\right) = (-1)^d \frac{d^n}{dt^n}\left(e^{-t^\alpha}\right)_{t=y}.$$

The function ϕ_d is elementary but the expressions are cumbersome for larger d. Using the representation (299) in (298), and inverting the order of integration by Fubini we get the density of $\mathbf{W}$ in integral form as

$$f_{\mathbf{W}}(\mathbf{w}) = \tag{208}$$

$$= \frac{\sqrt{\pi}\left(\frac{\beta}{a}\right)^{d/2-1}\left(\frac{\beta}{2}\right)^{d/2-1}}{(2\pi)^{d/2}\Gamma\left(\frac{d}{2} - \frac{1}{2}\right)\sqrt{|\mathbf{\Sigma}|}} \int_0^\infty \left[t(t+2)\right]^{d/2-3/2} \phi_d\left(\beta(1+t) - b/2\right) dt.$$

An application of the Cauchy-Schwarz inequality gives that $\beta \geq b/2$ for $\mathbf{w} \neq 0$ so that the argument in ϕ_d is always positive. The above representations involve elementary functions only and can be used to implement numerical maximum likelihood procedures. For $d = 2$ and $d = 3$ the expressions simplify considerably. We get

$$f_{\mathbf{W}}(\mathbf{w}) = \frac{1}{2\pi\sqrt{|\mathbf{\Sigma}|}} \int_0^\infty \frac{1}{\sqrt{t(t+2)}} \phi_2 \left(\beta(1+t) - \frac{b}{2} \right) dt \,. \tag{209}$$

for $d = 2$ and for $d = 3$ noting that $K_{1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}$

$$f_{\mathbf{W}}(\mathbf{w}) = \frac{1}{4\pi\sqrt{|\mathbf{\Sigma}|}} a^{-1/2} \phi_2(\beta - b/2) \tag{210}$$

which is a closed form expression. This is remarkable given that in general there is no closed form expression for f_S. Note that for odd d the functions $K_{d/2-1}$ are elementary and hence the densities can be expressed in closed form but the expressions are workable for smaller dimensions. These possibilities will be explored in subsequent research.

Despite the lack of explicit formulae for densities the multivariate asymmetric Weibull distributions can be simulated effectively. The remarkable algorithm due to Chambers, Mellow and Stuck (see [10]) modified by Weron (see [99]) provides a way to simulate stable random variables with parameter $\alpha \in (0, 1)$ easily. We outline the steps:

- generate a uniform random variable $X \sim U(0, \pi)$.
- substitute X into the function

$$U_\alpha(x) = \frac{[\sin(\alpha x)]^{\frac{\alpha}{1-\alpha}} \sin[(1-\alpha)x]}{(\sin x)^{\frac{1}{1-\alpha}}}$$

- generate $E \sim \exp(1)$.
- compute $S = \left[\frac{U_\alpha(x)}{E}\right]^{\frac{1-\alpha}{\alpha}}$. The random variable S has the standard stable distribution defined in (8).

Following the definition of the $W_d(\alpha, \mathbf{m}, \mathbf{\Sigma})$ distribution the simulation algorithm consists of the following steps:

- for given α generate a standard stable subordinator random variable S following the algorithm presented above.
- generate a standard exponential random variable E independent of S.

- Independently of S and E generate multivariate normal random vector $\mathbf{X} \sim \mathrm{N}_d(\mathbf{0}, \mathbf{\Sigma})$.
- Compute $\mathbf{W}$ using the representation (187).

Figure 3 shows the simulated sample points from various bivariate distributions in the case $d = 2$. These snapshots clearly show how the choice of parameters affects the nature of these distributions. The asymmetric pattern can be observed from the graphs.

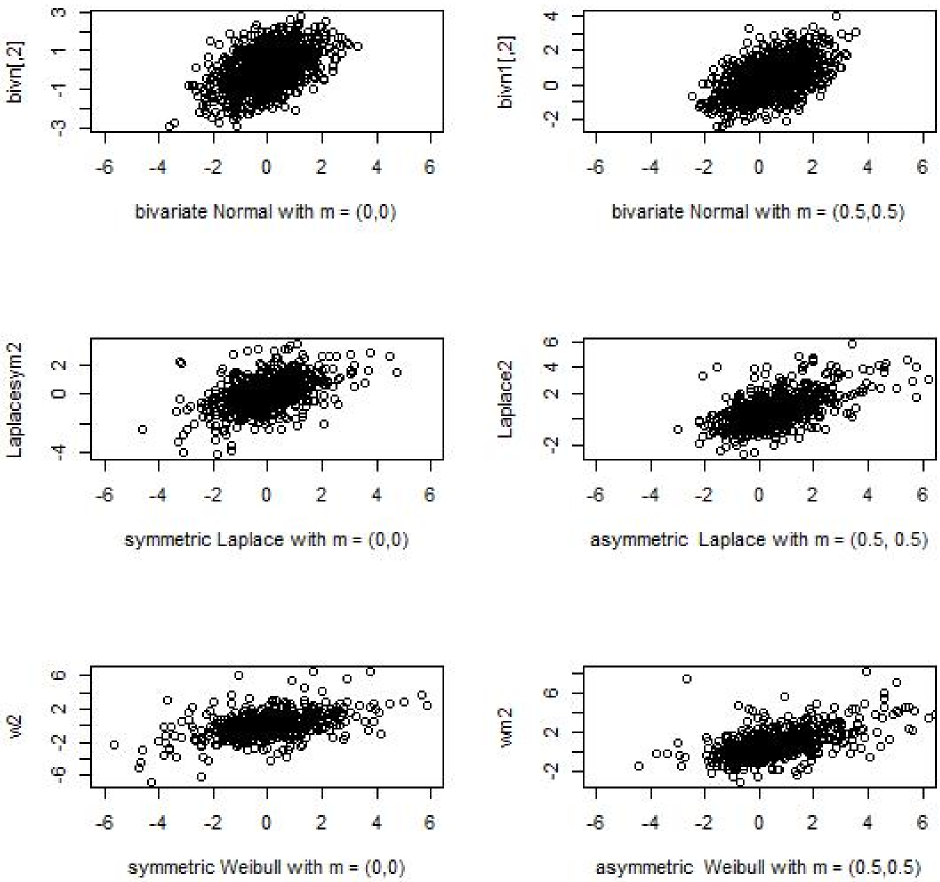

Figure 3: Scatterplots of various distributions

7 Quadrant probabilities in the bivariate case

Our objective here is to compute the probabilities of bivariate skew-Weibull random vector $\mathbf{W} = (W_1, W_2) \sim W_2(\alpha, \mathbf{m}, \mathbf{\Sigma})$ landing in the four quadrants,

$$\begin{aligned} Q_1 &= \{(w_1, w_2) : w_1 > 0, w_2 > 0\}, \quad Q_2 = \{(w_1, w_2) : w_1 < 0, w_2 > 0\}, \\ Q_3 &= \{(w_1, w_2) : w_1 < 0, w_2 < 0\}, \quad Q_4 = \{(w_1, w_2) : w_1 > 0, w_2 < 0\}. \end{aligned} \tag{211}$$

We shall assume that the parameter $\mathbf{\Sigma}$ is a non-singular, positive-definite matrix of the form

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \tag{212}$$

where $\sigma_1^2 = \mathbb{V}ar(X_1) > 0$, $\sigma_2^2 = \mathbb{V}ar(X_2) > 0$, while $\rho \in (-1, 1)$ is the correlation of X_1 and X_2, with $\mathbf{X} = (X_1, X_2)$ being an underlying mean-zero bivariate normal random vector in the stochastic representation (187) of $\mathbf{W}$. In this notation, the PDF of $\mathbf{X}$ is of the form

$$g(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{x_1^2}{\sigma_1^2} - 2\rho\frac{x_1x_2}{\sigma_1\sigma_2} + \frac{x_2^2}{\sigma_2^2}\right]\right\}. \tag{213}$$

First, we note that the stable subordinator S in the stochastic representation of $\mathbf{W}$, and the associated parameter α, play no role in these probabilities. Furthermore, it is clear that in the special case $\mathbf{m} = \mathbf{0}$, we have $\mathbb{P}(\mathbf{W} \in Q_i) = \mathbb{P}(\mathbf{X} \in Q_i)$. Since the latter probabilities are well-known [see, e.g., [101]], we have the following result.

Proposition 7.1 *If* $\mathbf{W} \sim W_2(\alpha, \mathbf{m}, \mathbf{\Sigma})$ *and* $\mathbf{m} = \mathbf{0}$, *then*

$$\mathbb{P}(\mathbf{W} \in Q_1) = \mathbb{P}(\mathbf{W} \in Q_3) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho \tag{214}$$

and

$$\mathbb{P}(\mathbf{W} \in Q_2) = \mathbb{P}(\mathbf{W} \in Q_4) = \frac{1}{4} - \frac{1}{2\pi}\sin^{-1}\rho. \tag{215}$$

Going beyond this special case, we shall first assume that the two components of the parameter $\mathbf{m} = (m_1, m_2)$ are both positive. This is reflected in our notation for the four probabilities,

$$P_{+,+}^{(i)} = \mathbb{P}(\mathbf{W} \in Q_i), \quad i = 1, 2, 3, 4, \tag{216}$$

with the subscript $+, +$ indicating $m_1 > 0$, $m_2 > 0$. As will be shown in the sequel, the other cases, involving various combinations of the signs of m_1 and m_2, can be deduced from this pivotal case.

Proposition 7.2 *Let* $\mathbf{W} \sim W_2(\alpha, \mathbf{m}, \mathbf{\Sigma})$ *where* $m_1, m_2 > 0$. *Then we have*

$$\begin{aligned} P^{(1)}_{+,+} &= \frac{\sqrt{1-\rho^2}}{2\pi} \cdot \Big\{ \int_0^{\pi/2} \frac{d\theta}{1-\rho\sin(2\theta)} + \int_0^{\pi/2} \frac{d\theta}{1+\rho\sin(2\theta)+4(\sigma_1/m_1)^2(1-\rho^2)\cos^2\theta} \\ &+ \int_0^{\pi/2} \frac{d\theta}{1+\rho\sin(2\theta)+4(\sigma_2/m_2)^2(1-\rho^2)\sin^2\theta} \\ &+ \int_0^{\pi/2} \frac{d\theta}{1-\rho\sin(2\theta)+4\max\{(\sigma_1/m_1)^2\cos^2\theta,(\sigma_2/m_2)^2\sin^2\theta\}(1-\rho^2)} \Big\}, \end{aligned} \tag{217}$$

$$\begin{aligned} P^{(2)}_{+,+} &= \frac{\sqrt{1-\rho^2}}{2\pi} \cdot \Big\{ \int_0^{\pi/2} \frac{d\theta}{1+\rho\sin(2\theta)} - \int_0^{\pi/2} \frac{d\theta}{1+\rho\sin(2\theta)+4(\sigma_1/m_1)^2(1-\rho^2)\cos^2\theta} \\ &+ \int_0^{\tan^{-1}\left(\frac{\sigma_1}{\sigma_2}\frac{m_2}{m_1}\right)} \frac{d\theta}{1-\rho\sin(2\theta)+4(\sigma_2/m_2)^2(1-\rho^2)\sin^2\theta} \\ &- \int_0^{\tan^{-1}\left(\frac{\sigma_1}{\sigma_2}\frac{m_2}{m_1}\right)} \frac{d\theta}{1-\rho\sin(2\theta)+4(\sigma_1/m_1)^2(1-\rho^2)\cos^2\theta} \Big\}, \end{aligned} \tag{218}$$

$$\begin{aligned} P^{(3)}_{+,+} &= \frac{\sqrt{1-\rho^2}}{2\pi} \cdot \Big\{ \int_0^{\pi/2} \frac{d\theta}{1-\rho\sin(2\theta)} \\ &- \int_0^{\pi/2} \frac{d\theta}{1-\rho\sin(2\theta)+4\min\{(\sigma_1/m_1)^2\cos^2\theta,(\sigma_2/m_2)^2\sin^2\theta\}(1-\rho^2)} \Big\}. \end{aligned} \tag{219}$$

$$\begin{aligned} P^{(4)}_{+,+} &= \frac{\sqrt{1-\rho^2}}{2\pi} \cdot \Big\{ \int_0^{\pi/2} \frac{d\theta}{1+\rho\sin(2\theta)} - \int_0^{\pi/2} \frac{d\theta}{1+\rho\sin(2\theta)+4(\sigma_2/m_2)^2(1-\rho^2)\sin^2\theta} \\ &+ \int_{\tan^{-1}\left(\frac{\sigma_1}{\sigma_2}\frac{m_2}{m_1}\right)}^{\pi/2} \frac{d\theta}{1-\rho\sin(2\theta)+4(\sigma_1/m_1)^2(1-\rho^2)\cos^2\theta} \\ &- \int_{\tan^{-1}\left(\frac{\sigma_1}{\sigma_2}\frac{m_2}{m_1}\right)}^{\pi/2} \frac{d\theta}{1-\rho\sin(2\theta)+4(\sigma_2/m_2)^2(1-\rho^2)\sin^2\theta} \Big\}. \end{aligned} \tag{220}$$

Proof. The result follows from standard conditioning argument, leading to

$$\begin{aligned} P^{(i)}_{+,+} &= \int_0^\infty \int_0^\infty P^{(i)}(x_1,x_2) g(x_1,x_2) dx_1 dx_2 \\ &+ \int_0^\infty \int_{-\infty}^0 P^{(i)}(x_1,x_2) g(x_1,x_2) dx_1 dx_2 \\ &+ \int_{-\infty}^0 \int_0^\infty P^{(i)}(x_1,x_2) g(x_1,x_2) dx_1 dx_2 \\ &+ \int_{-\infty}^0 \int_{-\infty}^0 P^{(i)}(x_1,x_2) g(x_1,x_2) dx_1 dx_2 \\ &= I + II + III + IV, \end{aligned} \tag{221}$$

where g is the bivariate normal density (213) and

$$P^{(i)}(x_1,x_2) = \mathbb{P}(\mathbf{W} \in Q_i | X_1 = x_1, X_2 = x_2), \;\; i = 1,2,3,4. \tag{222}$$

For example, with $i = 1$, the first integral in (221) reduces to

$$I = \int_0^\infty \int_0^\infty g(x_1,x_2) dx_1 dx_2, \tag{223}$$

as

$$P^{(1)}(x_1,x_2) = \mathbb{P}(m_1 E + \sqrt{2E} x_1 > 0, m_2 E + \sqrt{2E} x_2 > 0) = 1, \tag{224}$$

since $x_1, x_2 > 0$ and $m_1, m_2 > 0$. A substitution $x_1 = \sigma_1 y_1$, $x_2 = \sigma_2 y_2$, followed by switching to polar coordinates $y_1 = r\cos\theta$, $y_2 = r\sin\theta$, transforms (223) into

$$I = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_0^{\pi/2} \int_0^\infty r \exp\left\{ -\frac{(1-2\rho\sin\theta\cos\theta)r^2}{2(1-\rho^2)} \right\} dr d\theta. \tag{225}$$

Upon routine evaluation of the inner integral in (225), we arrive at

$$I = \frac{\sqrt{1-\rho^2}}{2\pi} \int_0^{\pi/2} \frac{d\theta}{1-\rho\sin(2\theta)}, \tag{226}$$

leading to the very first term in (217). We omit straightforward albeit quite tedious derivations of the remaining cases, which are obtained similarly.

Remark 5. Note that while the integrals in Proposition 7.2 generally do not seem to admit close forms (with the exception of the first one in each of the four expressions), they are relatively straightforward to compute viz. standard Monte Carlo integration. We also note that these probabilities are affected by the parameter ρ as well as the ratios σ_i/m_i, $i=1,2$. In the boundary cases $m_i=0$, any of the integrals that depend on m_i are evaluated by taking the limit as m_i approaches zero (and the majority of them reduce to zero). In particular, in the special case $m_1=m_2=0$, we have

$$P_{+,+}^{(1)} = P_{+,+}^{(3)} = \frac{\sqrt{1-\rho^2}}{2\pi} \int_0^{\pi/2} \frac{d\theta}{1-\rho\sin(2\theta)} \tag{227}$$

and

$$P_{+,+}^{(2)} = P_{+,+}^{(4)} = \frac{\sqrt{1-\rho^2}}{2\pi} \int_0^{\pi/2} \frac{d\theta}{1+\rho\sin(2\theta)}. \tag{228}$$

Moreover, the above integrals are explicit, since (227) and (228) coincide with (214) and (215), respectively.

As stated earlier, the other cases involving various combinations of the signs of m_1 and m_2 can be deduced from the pivotal case presented in Proposition 7.2. As we discuss this, we shall use a notation where we emphasize the dependence of the quadrants' probabilities on the parameters m_1, m_2, and ρ. That is, the quantity $P_{+,+}^{(i)}$ in (216) will be written as $P_{+,+}^{(i)}(m_1,m_2,\rho)$ to emphasize its dependence on these three parameters. Similarly, we shall write

$$P_{+,-}^{(i)}(m_1,m_2,\rho) = \mathbb{P}(\mathbf{W}\in Q_i), \quad i=1,2,3,4, \tag{229}$$

when we have $m_1>0$ and $m_2<0$,

$$P_{-,+}^{(i)}(m_1,m_2,\rho) = \mathbb{P}(\mathbf{W}\in Q_i), \quad i=1,2,3,4, \tag{230}$$

when we have $m_1<0$ and $m_2>0$, and

$$P_{-,-}^{(i)}(m_1,m_2,\rho) = \mathbb{P}(\mathbf{W}\in Q_i), \quad i=1,2,3,4, \tag{231}$$

when we have $m_1<0$ and $m_2<0$. Under this notation, we have the following result, where for brevity we write $P_{+,-}^{(i)}$, $P_{-,+}^{(i)}$, and $P_{-,-}^{(i)}$ to denote $P_{+,-}^{(i)}(m_1,m_2,\rho)$, $P_{-,+}^{(i)}(m_1,m_2,\rho)$, and $P_{-,-}^{(i)}(m_1,m_2,\rho)$, respectively.

Proposition 7.3 *In the above notation, the following relations hold:*

$$P^{(1)}_{+,-} = P^{(4)}_{+,+}(m_1, -m_2, -\rho), \;\; P^{(2)}_{+,-} = P^{(3)}_{+,+}(m_1, -m_2, -\rho), \tag{232}$$

$$P^{(3)}_{+,-} = P^{(2)}_{+,+}(m_1, -m_2, -\rho), \;\; P^{(4)}_{+,-} = P^{(1)}_{+,+}(m_1, -m_2, -\rho), \tag{233}$$

$$P^{(1)}_{-,+} = P^{(2)}_{+,+}(-m_1, m_2, -\rho), \;\; P^{(2)}_{-,+} = P^{(1)}_{+,+}(-m_1, m_2, -\rho), \tag{234}$$

$$P^{(3)}_{-,+} = P^{(4)}_{+,+}(-m_1, m_2, -\rho), \;\; P^{(4)}_{-,+} = P^{(3)}_{+,+}(-m_1, m_2, -\rho), \tag{235}$$

$$P^{(1)}_{-,-} = P^{(3)}_{+,+}(-m_1, -m_2, \rho), \;\; P^{(2)}_{-,-} = P^{(4)}_{+,+}(-m_1, -m_2, \rho), \tag{236}$$

$$P^{(3)}_{-,-} = P^{(1)}_{+,+}(-m_1, -m_2, \rho), \;\; P^{(4)}_{-,-} = P^{(2)}_{+,+}(-m_1, -m_2, \rho). \tag{237}$$

Proof. Since the above relations easily follow from the stochastic representation of $\mathbf{W}$, we shall omit them all but one, to illustrate the derivations. Consider the left-hand-side of the first relation in (232),

$$P^{(1)}_{+,-} = \mathbb{P}(\mathbf{W} \in Q_1) = \mathbb{P}(m_1 E + \sqrt{2E} X_1 > 0, m_2 E + \sqrt{2E} X_2 > 0), \tag{238}$$

where $m_1 > 0$, $m_2 < 0$, $E \sim \exp(1)$, and (X_1, X_2) is zero-mean bivariate normal with the covariance matrix (212). Clearly, the probability in (238) is the same as

$$\mathbb{P}(m_1 E + \sqrt{2E} X_1 > 0, (-m_2)E + \sqrt{2E}(-X_2) < 0), \tag{239}$$

where we have $m_1 > 0$, $-m_2 > 0$, E is as before, and $(X_1, -X_2)$ is zero-mean bivariate normal with the covariance matrix $\tilde{\mathbf{\Sigma}}$, which is the same as $\mathbf{\Sigma}$ given by (212) but with ρ replaced with $-\rho$. Thus, this probability coincides with the probability $\mathbb{P}(\tilde{\mathbf{W}} \in Q_4)$, where $\tilde{\mathbf{W}}$ is skew Weibull with the stochastic representation $\tilde{\mathbf{W}} = (m_1 E + \sqrt{2E} X_1, (-m_2)E + \sqrt{2E}(-X_2))$. Consequently, according to our notation, this is precisely the expression on the right-hand side of the first relation in (232). The other relations in (232) - (237) are established in a similar way.

8 Estimation of the parameters - multivariate case

In this chapter we derive the mean, the covariance, and estimation of the parameters based on the methods of moments. We focus on estimating the parameters α, m_1, m_2 and the coefficients of 2×2 matrix $\mathbf{\Sigma}$. In order to simplify the calculation, in the form $m\frac{E}{S} + \frac{\sqrt{E}}{S}X$, number 2 is omitted under the square root. Using the parametrization (13), the following relation can be obtained

$$\mathbb{E}(W_i W_j) = \mathbb{E}\Big[(m_i \frac{E}{S} + \frac{\sqrt{E}}{S} X_i)(m_j \frac{E}{S} + \frac{\sqrt{E}}{S} X_j)\Big]. \tag{240}$$

Taking into account that $\mathbb{E}(E^2) = 2$, $\mathbb{E}(S^{-2}) = \frac{1}{2}\Gamma(1 + \frac{2}{\alpha})$ and $\mathbb{E}(X_i) = 0$, the equation reduces to

$$\mathbb{E}(W_i W_j) = m_i m_j \Gamma(1 + \frac{2}{\alpha}) + \frac{\sigma_{ij}}{2}\Gamma(1 + \frac{2}{\alpha}). \tag{241}$$

In particular,

$$\mathbb{E}(W_i^2) = m_i^2 \Gamma\left(1 + \frac{2}{\alpha}\right) + \frac{\sigma_{ii}}{2}\Gamma\left(1 + \frac{2}{\alpha}\right). \tag{242}$$

where $\sigma_{i,i}$ denotes $Var(X_i)$. Furthermore,

$$\mathbb{E}(W_i) = m_i \Gamma\left(1 + \frac{1}{\alpha}\right). \tag{243}$$

In a similar way, the covariance is obtained as

$$Cov(W_i, W_j) = \Gamma\left(1 + \frac{2}{\alpha}\right) m_i m_j + \frac{1}{2}\Gamma\left(1 + \frac{2}{\alpha}\right)\sigma_{ij} - \Gamma^2\left(1 + \frac{1}{\alpha}\right) m_i m_j, \tag{244}$$

leading to

$$Cov(W_i, W_j) = \left[\frac{2}{\alpha}\Gamma\Big(\frac{2}{\alpha}\Big) - \Big(\frac{1}{\alpha}\Big)^2 \Gamma^2\Big(\frac{1}{\alpha}\Big)\right] m_i m_j + \frac{1}{\alpha}\Gamma\Big(\frac{2}{\alpha}\Big)\sigma_{ij}. \tag{245}$$

Using the multivariate notation and denoting $c_\alpha = \frac{2}{\alpha}\Gamma(\frac{2}{\alpha}) - (\frac{1}{\alpha})^2\Gamma^2(\frac{1}{\alpha})$ and $d_\alpha = \frac{1}{\alpha}\Gamma(\frac{2}{\alpha})$, the above equation reduces to:

$$Cov(\mathbf{W}) = \mathbf{m}\mathbf{m}' c_\alpha + \mathbf{\Sigma} d_\alpha. \tag{246}$$

As mentioned before, the estimation of the parameters α, m_1, m_2 as well as the coefficients of the 2×2 matrix $\mathbf{\Sigma}$ will be obtained by the method of moments. Only the bivariate case will be considered. The following equations are included in the calculation:

$$\bar{W}_1 = \frac{1}{n}\sum_{i=1}^{n} W_{1i} \approx \mathbb{E}(W_1) = m_1\Gamma(1+\frac{1}{\alpha}), \tag{247}$$

$$\bar{W}_2 = \frac{1}{n}\sum_{i=1}^{n} W_{2i} \approx \mathbb{E}(W_2) = m_2\Gamma(1+\frac{1}{\alpha}), \tag{248}$$

$$\overline{W_1^2} = \frac{1}{n}\sum_{i=1}^{n} W_{1i}^2 \approx \mathbb{E}(W_1^2) = m_1^2\Gamma(1+\frac{2}{\alpha}) + \frac{1}{2}\Gamma(1+\frac{2}{\alpha})\sigma_{11}, \tag{249}$$

$$\overline{W_2^2} = \frac{1}{n}\sum_{i=1}^{n} W_{2i}^2 \approx \mathbb{E}(W_2^2) = m_2^2\Gamma(1+\frac{2}{\alpha}) + \frac{1}{2}\Gamma(1+\frac{2}{\alpha})\sigma_{22}, \tag{250}$$

$$\overline{W_1W_2} = \frac{1}{n}\sum_{i=1}^{n} W_{1i}W_{2i} \approx \mathbb{E}(W_1W_2) = m_1m_2\Gamma(1+\frac{2}{\alpha}) + \frac{1}{2}\Gamma(1+\frac{2}{\alpha})\sigma_{12}, \tag{251}$$

$$\frac{1}{n}\sum_{i=1}^{n} W_{1i}^2W_{2i}^2 \approx \mathbb{E}(W_1^2W_2^2) = \Gamma(1+\frac{4}{\alpha})$$
$$\Big\{m_1^2m_2^2 + \frac{1}{4}[m_1^2\sigma_{22} + m_2^2\sigma_{11}] + \frac{1}{6}\sigma_{12}^2 + \frac{1}{12}\sigma_{11}\sigma_{22} + m_1m_2\sigma_{12}\Big\}. \tag{252}$$

The latter formula is computed using the following proposition:

Proposition 8.1 *Let $W \sim W_d(\alpha, \mathbf{m}, \mathbf{\Sigma})$. Then the following formula applies:*

$$\mathbb{E}(W_1^2W_2^2) = \Gamma(1+\frac{4}{\alpha})\Big\{m_1^2m_2^2 + \frac{1}{4}[m_1^2\sigma_{22} + m_2^2\sigma_{11}] + \frac{1}{6}\sigma_{12}^2 + \frac{1}{12}\sigma_{11}\sigma_{22} + m_1m_2\sigma_{12}\Big\}.$$

Proof: We start with

$$\mathbb{E}\Bigg\{\Big(\frac{m_1E + \sqrt{E}X_1}{S}\Big)^2\Big(\frac{m_2E + \sqrt{E}X_2}{S}\Big)^2\Bigg\} = \tag{253}$$

$$= \Big[m_1^2m_2^2\mathbb{E}(E^4) + m_1^2\mathbb{E}(E^3)\mathbb{E}(X_2^2) + m_2^2\mathbb{E}(E^3)\mathbb{E}(X_1^2) + \mathbb{E}(E^2)\mathbb{E}(X_1^2X_2^2) +$$
$$+ 2m_2\mathbb{E}(E^{\frac{5}{2}})\mathbb{E}(X_1^2X_2) + 2m_1\mathbb{E}(E^{\frac{5}{2}})\mathbb{E}(X_1X_2^2) + 4m_1m_2\mathbb{E}(E^3)\mathbb{E}(X_1X_2)\Big]\mathbb{E}(S^{-4})$$

and taking into account that $\mathbb{E}(X_1X_2) = \sigma_{12}$, $\mathbb{E}(E^n) = \Gamma(1+n)$ and

$$\mathbb{E}(X_1^2X_2) = 0, \;\; \mathbb{E}(X_1X_2^2) = 0, \;\; \mathbb{E}(X_1^2X_2^2) = 2\sigma_{12}^2 + \sigma_{11}\sigma_{22}, \tag{254}$$

(see [68]), leads to (252). The following notation $\sigma^2_{X_1} = \sigma_{11}$, $\sigma^2_{X_2} = \sigma_{22}$, $\rho^2 = \frac{\sigma^2_{12}}{\sigma_{11}\sigma_{22}}$, $\sigma_{11} = \sigma^2_x$, $\sigma_{22} = \sigma^2_y$, and $\sigma_{12} = \rho\sigma_x\sigma_y$, where ρ is a corelation coefficient of X_1 and X_2 will be used. It remains to find

$$\mathbb{E}(X_1^n X_2^m) = \sigma^n_{X_1}\sigma^m_{X_2}\sum_{i=0}^{n}\binom{n}{i}(1-\rho^2)^{\frac{i}{2}}\gamma_i\rho^{n-i}\gamma_{m+n-i}, \tag{255}$$

where $\mathbb{E}(Z^i) = \gamma_i$, Z_i is a standard normal random variable and

$$\begin{cases} \gamma_i = 0 \text{ if } i \text{ is odd} \\ \gamma_i = (i-1)!! \text{ if } i \text{ is even.} \end{cases} \tag{256}$$

Here, $(i-1)!!$ denotes the double factorial, that is, the product of every odd number from $i-1$ to 1.

Proposition 8.2 *If* $\mathbf{X} \sim \mathrm{N}_d(0, \mathbf{\Sigma})$, *the following result holds:*

$$\mathbb{E}(X_1^n X_2^m) = \sigma^n_{X_1}\sigma^m_{X_2}\sum_{i=0}^{n}\binom{n}{i}(1-\rho^2)^{\frac{i}{2}}\gamma_i\rho^{n-i}\gamma_{m+n-i},$$

Proof: It is known

$$\mathbb{E}(X_1^n X_2^m) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^m \exp\left[-\frac{1}{2(1-\rho^2)}\Big(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}-\frac{2\rho xy}{\sigma_x\sigma_y}\Big)\right]dxdy. \tag{257}$$

Substituting $\frac{x}{\sigma_x} = u$ and extending the expression $u^2 - \frac{2\rho uy}{\sigma_y}$ to complete the square, the following form transforming the inner integral is obtained,

$$\sigma_x^{n+1}y^m \exp\left(-\frac{y^2}{2\sigma_y^2}\right)\int_{-\infty}^{\infty} u^n \exp\left(-\frac{1}{2(1-\rho^2)}(u-\frac{\rho y}{\sigma_y})^2\right)du, \tag{258}$$

which after some adjustments leads to:

$$\sqrt{2\pi}\sqrt{1-\rho^2}\sigma_x^{n+1}y^m\exp\left(-\frac{y^2}{2\sigma_y^2}\right)\mathbb{E}(U^n). \tag{259}$$

Since $U \sim N(\frac{\rho y}{\sigma_y}, 1-\rho^2)$, it follows that $u = \sqrt{1-\rho^2}Z + \frac{\rho y}{\sigma_y}$ where $Z \sim N(0,1)$. Taking the $n-th$ power of $\sqrt{1-\rho^2}Z + \frac{\rho y}{\sigma_y}$, we have the following form:

$$\sqrt{2\pi}\sqrt{1-\rho^2}\sigma_x^{n+1}y^m\exp\left(-\frac{y^2}{2\sigma_y^2}\right)\mathbb{E}\left(\sum_{i=0}^{n}\binom{n}{i}\sqrt{1-\rho^2}^{\,i}Z^i(\frac{\rho y}{\sigma_y})^{n-i}\right). \tag{260}$$

Applying the expectation, and substituting $\mathbb{E}(Z^i) = \gamma_i$, after some constant reductions, the outer integral becomes:

$$\frac{\sigma_x^n}{\sqrt{2\pi}\sigma_y}\int_{-\infty}^{\infty} y^m \exp\left(-\frac{y^2}{2\sigma_y^2}\right)\left(\sum_{i=0}^{n}\binom{n}{i}(1-\rho^2)^{\frac{i}{2}}\gamma_i\rho^{n-i}(\frac{y}{\sigma_y})^{n-i}\right)dy. \tag{261}$$

The above equation can be written as

$$\frac{\sigma_x^n}{\sqrt{2\pi}\sigma_y}\sum_{i=0}^{n}\binom{n}{i}(1-\rho^2)^{\frac{i}{2}}\gamma_i\rho^{n-i}\sigma_y^m\int_{-\infty}^{\infty}\left(\frac{y}{\sigma_y}\right)^{m+n-i}\exp\left(-\frac{1}{2}\left(\frac{y}{\sigma_y}\right)^2\right)dy, \tag{262}$$

which, after substitution $z = \frac{y}{\sigma_y}$, leads to

$$\sigma_x^n\sigma_y^m\sum_{i=0}^{n}\binom{n}{i}(1-\rho^2)^{\frac{i}{2}}\gamma_i\rho^{n-i}\int_{-\infty}^{\infty} z^{m+n-i}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}z^2\right)dz. \tag{263}$$

The latter integral defines $E(Z^{m+n-i}) = \gamma_{m+n-i}$ which gives us the final result stated in (255).

To estimate the parameters, m_1 and m_2 will be expressed from (247) and (248) yielding:

$$m_1 = \frac{\overline{W_1}}{\Gamma(1+\frac{1}{\alpha})} \quad \text{and} \quad m_2 = \frac{\overline{W_2}}{\Gamma(1+\frac{1}{\alpha})}. \tag{264}$$

Substituting into (249) - (251) and denoting

$$G_\alpha = \frac{\Gamma(1+\frac{2}{\alpha})}{\Gamma^2(1+\frac{1}{\hat{\alpha}})}, \tag{265}$$

the following equations arise:

$$\hat{\sigma}_{11} = \frac{2\left[\overline{W_1^2} - G_\alpha(\overline{W_1})^2\right]}{\Gamma(1+\frac{2}{\hat{\alpha}})}, \tag{266}$$

$$\hat{\sigma}_{22} = \frac{2\left[\overline{W_2^2} - G_\alpha(\overline{W_2})^2\right]}{\Gamma(1+\frac{2}{\hat{\alpha}})}, \tag{267}$$

$$\hat{\sigma}_{12} = \frac{2\left[\overline{W_1W_2} - G_\alpha\overline{W_1}\ \overline{W_2}\right]}{\Gamma(1+\frac{2}{\hat{\alpha}})}. \tag{268}$$

Substituting these expressions into (252) and some lenghty calculation, the following equation depending only on the parameter α is obtained:

$$E(W_1^2W_2^2) = \Gamma(1+\frac{4}{\alpha})\Bigg\{\frac{(\overline{W}_1\,\overline{W}_2)^2}{\Gamma^4(1+\frac{1}{\alpha})}+\frac{1}{2}\frac{(\overline{W}_1)^2\left[\overline{W_2^2}-G_\alpha(\overline{W}_2)^2\right]+(\overline{W}_2)^2\left[\overline{W_1^2}-G_\alpha(\overline{W}_1)^2\right]}{\Gamma^2(1+\frac{1}{\alpha})\Gamma(1+\frac{2}{\alpha})}$$

$$+\frac{2}{3}\frac{\left[\overline{W_1W_2}-G_\alpha\overline{W}_1\,\overline{W}_2\right]^2}{\Gamma^2(1+\frac{2}{\alpha})}$$

$$+\frac{1}{3}\frac{\left[\overline{W_1^2}-G_\alpha(\overline{W}_1)^2\right]\left[\overline{W_2^2}-G_\alpha(\overline{W}_2)^2\right]}{\Gamma^2(1+\frac{2}{\alpha})}+\frac{2\overline{W}_1\overline{W}_2\left[(\overline{W_1W_2})-G_\alpha\overline{W}_1\,\overline{W}_2\right]}{\Gamma^2(1+\frac{1}{\alpha})\Gamma(1+\frac{2}{\alpha})}\Bigg\}, \quad (269)$$

where

$$G_\alpha = \frac{\Gamma(1+\frac{2}{\alpha})}{\Gamma^2(1+\frac{1}{\alpha})}. \quad (270)$$

In order to show the importance of the estimation procedure, numerical search is applied. To estimate parameter α, formula (269) is coded in statistical package R (see Appendices). Since the equation depends only on the parameter α, the bisection method is performed (see Appendices) yielding the result of $\alpha = 0.8419174$.Standard errors are estimated by a bootstrap study: once the parameters are estimated new samples are generated from the asymmetric Weibull distribution with the estimated parameters using the simulation algorithms (described previously in Chapter 6) and parameters are then reestimated. After many repetitions one can give standard errors from the empirical sampling distributions. Applying the above formulas and procedures, the estimates of the parameters are obtained and presented in Table 4.

9 Application - multivariate case

This section presents the asymmetric multivariate double Weibull model of currency exchange rates.Various distributions were used in the univariate case in the past, including stable Paretian laws (see Westerfield [100], McFarland et al. [72, 73], So [89], Koedjik et al. [47] and Nolan [77]), Student-t distribution (see Boothe and Glassman [8], Koedjik et al. [47]), mixture of normals (see Boothe and Glassman [8], Tucker and Pond [94]), asymmetric Laplace (see Kozubowski and Podgórski [52]), exponential power (see Ayebo and Kozubowski [6]), and (double) Weibull distribution (see Chenyao et al. [9]). Still, there is no general consensus regarding the best theoretical model, even though Chenyao et al. (see [9]) found the fit of the double Weibull model to be the best. Jurić and Kozubowski, (see [42]) showed that the skew univariate Weibull model outperformed the "competitors" .
Following Chenyao (see [9]), Hürliman (see [34]) and Mittnik and Rachev (see [74]), who report excellent results with the (double) Weibull distribution, the fit of the second representation of *asymmetric* double Weibull model to currency exchange rates, comparing the fit to that of normal, asymmetric Laplace (AL), and exponential power distributions (EP) showed the best results (see Jurić and Kozubowski [43]).
The same approach from the univariate case has been performed in the multivariate case. Financial institutions, however, have to deal with risks arising from baskets of currencies whose exchange rate movements exhibit heavy tails, correlations and asymmetries. We model such multivarite exchange rate data by the multivariate asymmetric Weibull distribution. The parameters are estimated using the method of moments and goodness of fit checks show a promising fit. A particular advantage of the multivariate asymmetric Weibull distribution is that by Corollary 5.1 (ii) linear combinations of the components have one dimensional asymmetric Weibull distributions. Baskets of currencies are exactly such linear combinations which means that the model can be used to assess the risks of arbitrary baskets of currencies.
The new data set contains daily currency exchange rates for transforming USD to JPY and GBP to JPY covering the period from June, 1st, 2000 until, May 30th 2014. The variable of interest is the logarithm of the exchange rate ratio for two consecutive days, and the data were transformed accordingly, resulting in 3652 values. The data set was imported from the web site http://www.global-view.com/forex-trading-tools/forex-history/.

For the chosen dataset the estimates are as follows with standard errors in parentheses:

$$\hat{p} = 0.0197(0.0023) \qquad \hat{\alpha} = 0.8419(0.0438)$$

and

$$\hat{\mathbf{m}} = 10^{-4}\begin{pmatrix} -0.2021 & 0.1686 \\ (0.9860) & (1.2157) \end{pmatrix} \quad \hat{\mathbf{\Sigma}} = 10^{-4}\begin{pmatrix} 0.1434 & 0.1185 \\ (0.0178) & (0.0142) \\ 0.1185 & 0.2167 \\ (0.0142) & (0.268) \end{pmatrix}$$

The results are summarized in the following table:

$m_1 = -0.00002021$	$m_2 = 0.00001686$
$\sigma_{11} = 0.00001434$	$\sigma_{22} = 0.00002167$
$\sigma_{12} = 0.00001185$	$\alpha = 0.8419174$

Table 4: Estimated values of m_1, m_2, σ_{11}, σ_{22}, σ_{12} and α obtained from the data set imported from http://www.global-view.com/forex-trading-tools/forex-history/

The scatterplot in Figure 4 clearly shows the asymmetric pattern of the data. A glance at simulated scatterplots in Figure 3 indicates that the multivariate asymmetric Weibull distribution appears to be a good model for the exchange rate data.

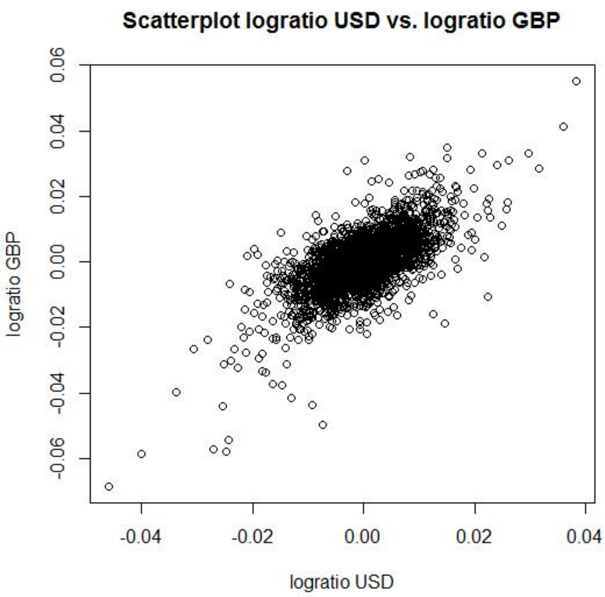

Figure 4: Scatterplot showing log-returns USD/JPY vs. log-returns GBP/JPY

The following results presented in the histograms for transforming USD to JPY and GBP to JPY along with the QQ plots are obtained.

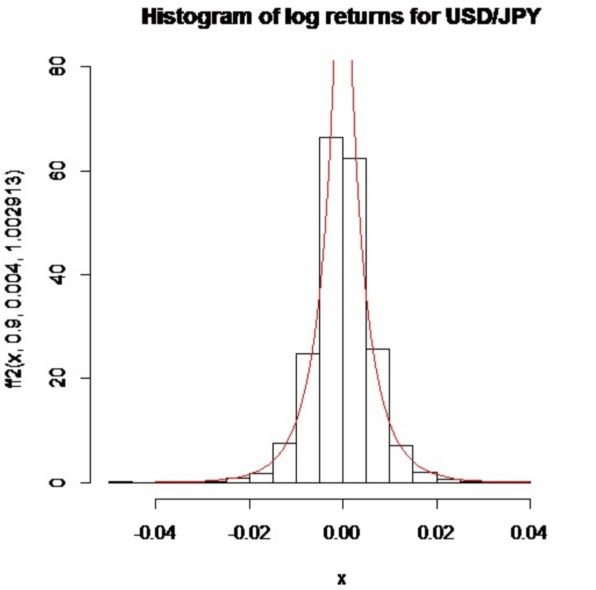

Figure 5: Histogram of log-returns transforming USD to JPY

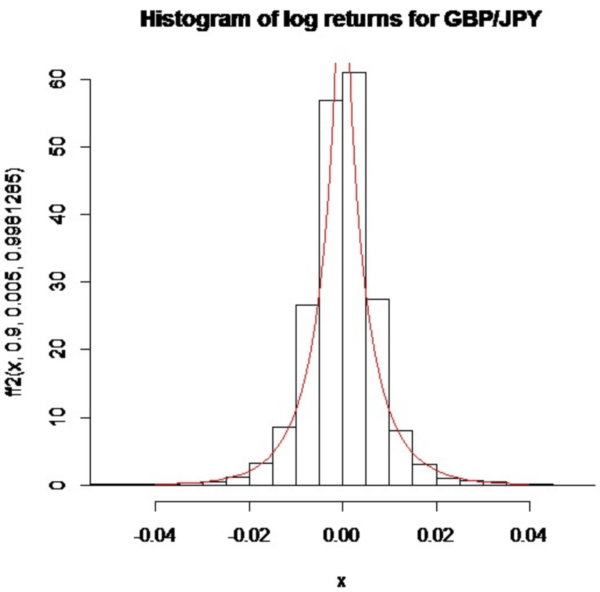

Figure 6: Histogram of log-returns transforming GBP to JPY

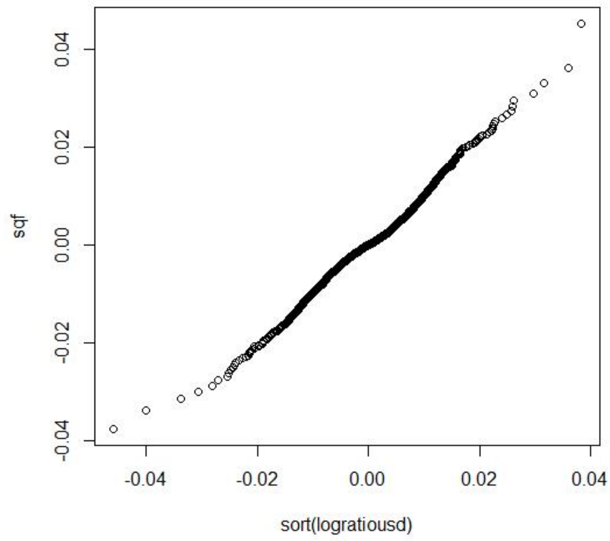

Figure 7: QQ plot comparing log-returns of USD/JPY

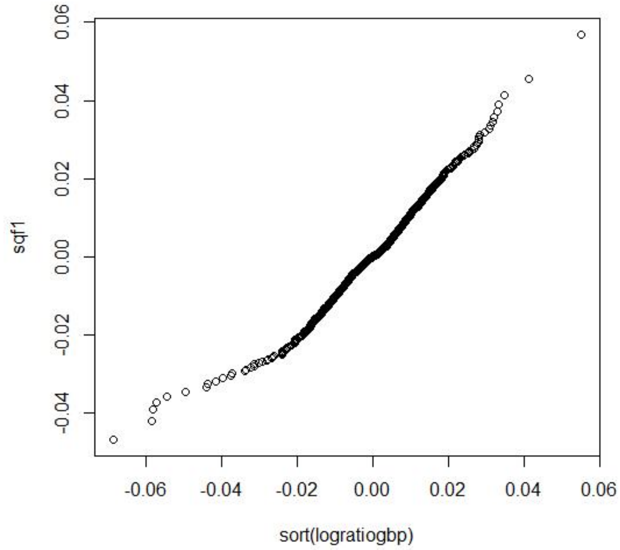

Figure 8: QQ plot comparing log-returns of GBP/JPY

These charts above show that the fit of QQ plot comparing log returns of GBP/JPY is reasonably good, but not as good as one for log returns of USD/JPY. The explanation may be the fact that the GBP dataset exhibits more variability than the USD data set. To show that the asymmetric Weibull distribution still has the modeling potential, the histograms and QQ plots comparing the normal distribution with the data are created. The results are shown in the following graphical presentations.

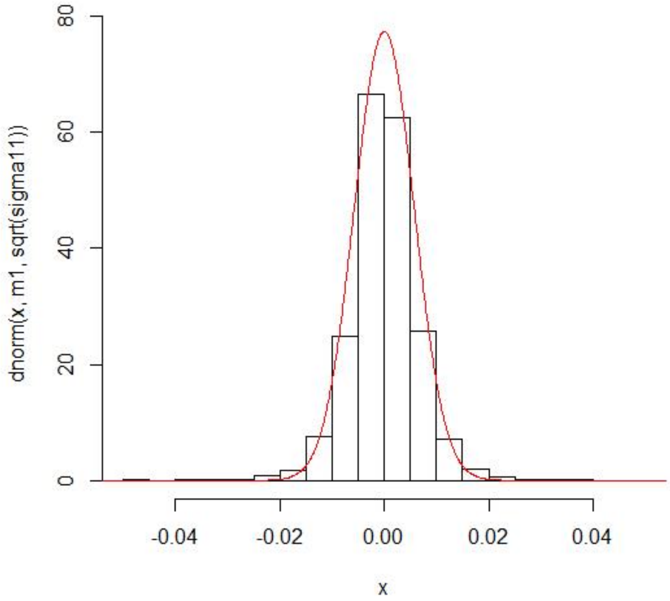

Figure 9: Histogram of log-returns transforming USD to JPY, comparison to normal distribution

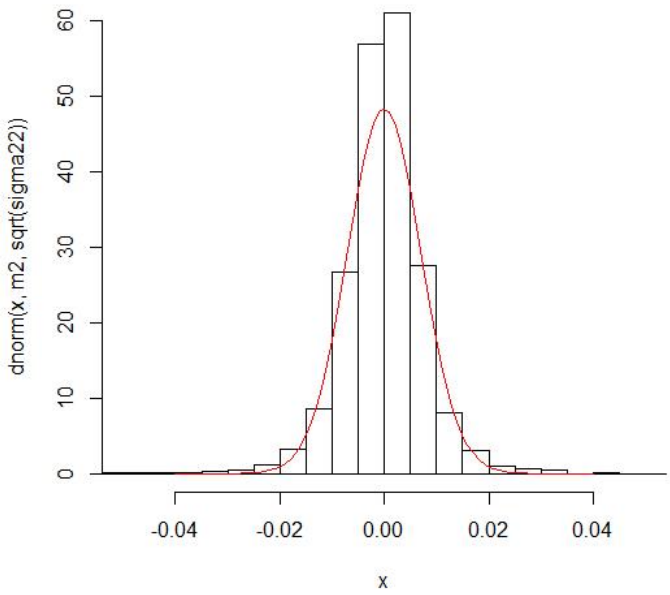

Figure 10: Histogram of log-returns transforming GBP to JPY, comparison to normal distribution

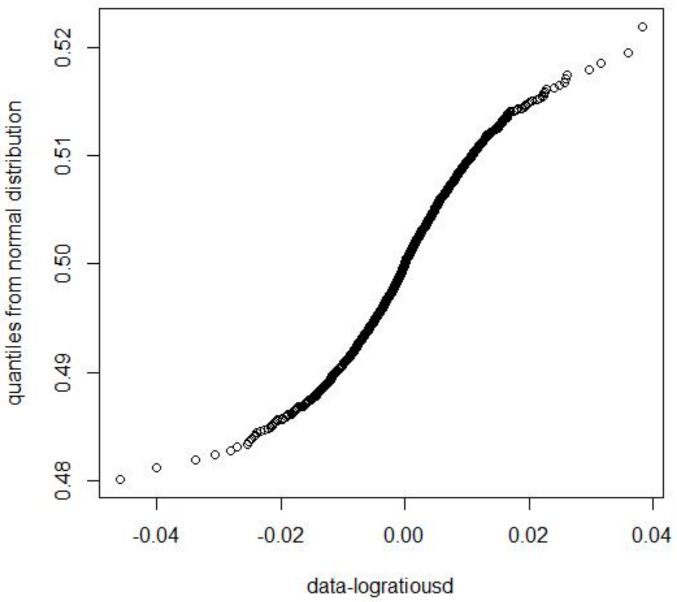

Figure 11: Normal QQ plot comparing log- returns of USD/JPY

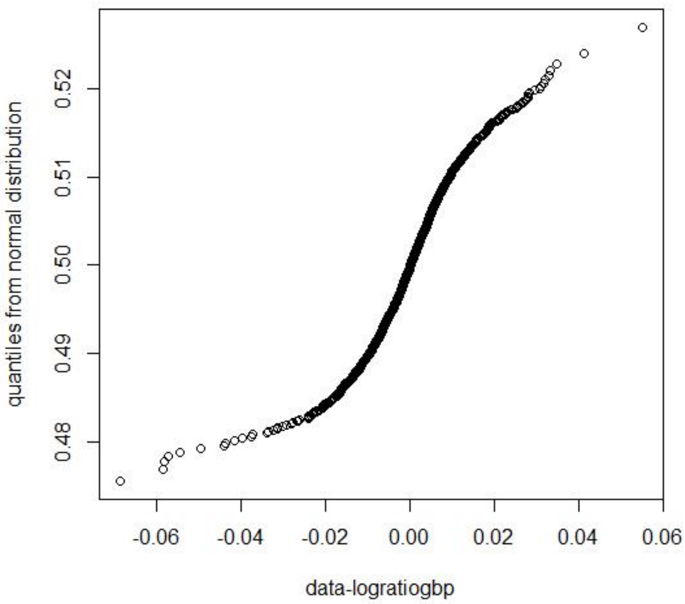

Figure 12: Normal QQ plot comparing log - returns of GBP/JPY

Obviously, QQ plots comparing the data with normal distribution did not show a good fit proving that multivariate Weibull distribution fits the data in a more suitable way. In order to show that multivariate Weibull distribution performs a good fit, projections in the directions of the several vectors are chosen. The parameters in the new univariate data sets are estimated using the method of moments in the same way as with previous univariate data transforming USD to JPY and GBP to JPY. By Corollary 5.1, the projections are univariate asymmetric Weibull distributions. We can check the goodness of fit by looking at QQ plots for projections. The similar comparison was presented in Dhar et al. (see [14]). Obtained QQ plots along with analogous fits of the normal distributions are presented in the following graphical presentation. It can be clearly seen that fits of multivariate Weibull distribution showed a better fit than QQ plots showing the comparison with normal distribution.

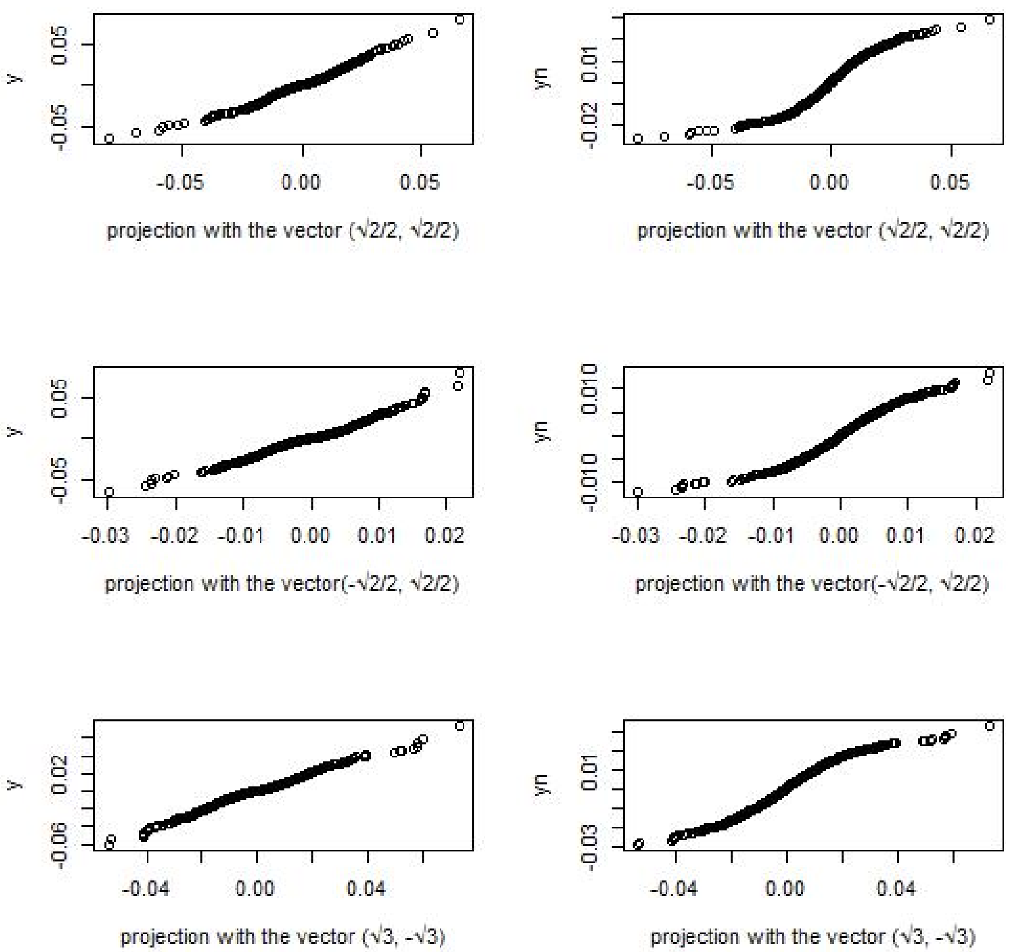

Figure 13: QQ plot comparing log-returns of the Weibull (on the left) and normal distribution (on the fight) in the direction of the chosen vectors

In addition, three-dimensional histogram of log-returns of the USD/JPY and GBP/JPY data is presented. It can be clearly seen that the data presented exhibit the same pattern as those in the two dimensional scatterplots.

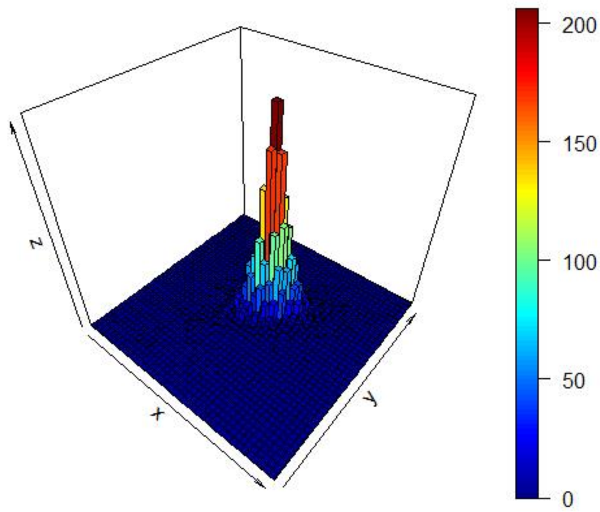

Figure 14: Three-dimensional histogram including 40 log- returns of the USD/JPY and GBP/JPY

Also, it can be observed that just as in the univaritae case, the bivariate data set does not imply a significant asymmetry which can be seen by noting that the parameters m are close to zero. Still this can be interpreted as statistical evidence that it may be necessary to model asymmetry.

10 Conclusion

Properties of the classical Weibull distribution are reviewed at the beginning of the work pointing out some important facts connected with stability and limiting properties. We summarized some basic facts about the symmetric (double) univariate Weibull distribution and introduce asymmetric double Weibull distributions along with their basic properties. Maximum likelihood method is used to estimate the parameters. Some estimators are available in closed form, while others require a numerical search.

In addition, different variations of, both symmetric and asymmetric multivariate and univariate Weibull distributions are considered. Different models are considered, each supported with p.d.f.'s c.d.f.'s, moment or maximum likelihood estimation and application part.

Next, introduction of multivariate extension of the one dimensional asymmetric Weibull distribution to multivariate setting is presented finding a natural way to extend the univariate Weibull distribution to $\mathbb{R}^d$. The method of moments is used to estimate parameters. Extending applications of asymmetric univariate case to multivariate setting was performed generalizing the findings about the usefulness of asymmetric multivariate distribution in the area of finance.

We believe that this work is a contribution to the science. The application of asymmetric double Weibull distribution of type II (in univariate case) indicates the usefulness of this model in mathematical finance. It is shown that the comparison with other models proves that this model exhibits a good fit for many types of financial data.

This is an indication of the modeling potential of these distributions in multivariate setting. Given the potential for applications it is necessary to examine and establish a firm theoretical background. The dissertation develops theoretical properties and estimation procedures for this new family of distributions. An application to two-dimensional currency exchange data set is examined. It exhibits a good fit and captures the asymmetry presented in the data.

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Appendices

LIST OF APPENDICES

APPENDIX A: ESTIMATION - QQ PLOTS-HISTOGRAMS

```
d<-read.table("fdata.txt")
logratiousd<-d[,1]
logratiogbp<-d[,2]
A2<-mean(logratiousd)
B2<-mean(logratiogbp)
C2<-mean(logratiousd*logratiogbp)
K2<-mean(logratiousd^2)
L2<-mean(logratiogbp^2)
f4<-function(alpha)
{f4<-gamma(4/alpha +1)
f4}
f1<-function(alpha)
{f1<- f1}
f2<-function(alpha)
{f2<-gamma(2/alpha +1)
f2}
G<-function(alpha)
{G<-1/gamma(1+1/alpha)^2
G}
f1sq<-function(alpha)
{f1sq<-gamma(1/alpha +1)^2
f1sq}
f1fth<-function(alpha)
{f1fth<-gamma(1/alpha +1)^4
f1fth}
f2sq<-function(alpha)
{f2sq<-gamma(2/alpha +1)^2
f2sq}
p11<-function(alpha){
p11<-(A2*B2)^2/f1fth(alpha)
p11}
p22<-function(alpha){
p22<-1/2*(A2^2*(L2-G(alpha)*B2^2)+B2^2*(K2-G(alpha)*
A2^2))/(f1sq(alpha)*f2(alpha))
p22}
p33<-function(alpha){
p33<-2/3*(C2-G(alpha)*A2*B2)^2
/f2sq(alpha)
p33}
p44<-function(alpha){
p44<-1/3*(K2-G(alpha)*A2^2)*(L2-G(alpha)*
B2^2)/f2sq(alpha)
p44}
p55<-function(alpha){
```

```
p55<-2*(A2*B2)*(C2-G(alpha)*A2*B2)/(f1sq(alpha)*
f2(alpha))
p55}
EST<-function(alpha){
EST<-f4(alpha)*(p11(alpha)+p22(alpha)+p33(alpha)+
p44(alpha)+p55(alpha))-mean(logratiousd^2*logratiogbp^2)
EST}

#Bisection -Tolerance:epsilon=0.0000000001.
epsilon <- 0.0000000001
left<-0.5
right<-1.0
difference<-1
while ( difference > epsilon) {
currentestimate<-(left+right)/2
if( EST(currentestimate) >0 ) {
left<-currentestimate
}
else
{
right<-currentestimate
}
difference<-right-left
}
alphahat<-currentestimate
alphahat = 0.8419174

#Estimates of the parameters
m1hat<-A2/f1(alphahat)
m2hat<-B2/f1(alphahat)
sigma11hat<-K2/f2(alphahat)-m1hat^2
sigma22hat<-L2/f2(alphahat)-m2hat^2
sigma12hat<-C2/f2(alphahat)-m1hat*m2hat
gamma(1/alphahat +1)
m1hat = -2.021262e-05
m2hat =  1.685584e-05
sigma11hat = 1.434195e-05
sigma22hat = 2.166902e-05
sigma12hat = 1.184681e-05

#USD/JPY values
alphahat = 0.8419174
sigmahatusd<-sqrt(sigma11hat/2)
kappa1hat<-(sqrt(4*sigma11hat)+m1hat^2-m1hat)/
(2*sqrt(sigma11hat))
sigmahatusd= 0.002677868          #sigma for usd
```

```
kappa1hat= 1.002669                  #kappa for usd

#GBP/JPY values
alphahat = 0.8419174
m1hat<-A2/gamma(1+1/alphahat)
m2hat<-B2/gamma(1+1/alphahat)
sigma11hat<-2*(K2-(G(alphahat)*A2^2))/f2(alphahat)
sigma22hat<-2*(L2-(G(alphahat)*B2^2))/f2(alphahat)
sigma12hat<-2*(C2-(G(alphahat)*B2*A2))/f2(alphahat)
sigmahatgbp<-sqrt(sigma22hat/2)
kappa2hat<-(sqrt(4*sigma22hat+m2hat^2)-m2hat)
/(2*sqrt(sigma22hat))
sigmahatgbp=0.0046550          #sigma for gbp
kappa2hat=0.9987206              #kappa for gbp

#PDF,USD/JPY
alphahat=0.8419174,sigmahatusd=0.002677868,
kappa1hat= 1.002669
ff2<-function(x,a,s,k)
{
x<-seq(-0.04,0.04, 0.001)
y<-rep(1,times=length(x))
for(i in 1:length(x)) {
if (x[i]>=0)
{y[i]<- ((1/s^a)*(a*k))/(1+k^2)*(x[i]*k)^(a-1)
*exp(-(k*x[i]/s)^a)}
else
{y[i]<-((1/s^a)*(a*k))/(1+k^2)*(-x[i]/k)^(a-1)
*exp(-(-x[i]/(k*s))^a)}
 y[i]
 }
 y
 }
x<-seq(-0.04,0.04,0.001)
hist(logratiousd,probability="TRUE",xlim=c(-0.05,0.05),
ylim=c(0,78),xlab=NULL,ylab=NULL)
par(new=T)
plot(x,ff2(x,0.8419174,0.002677868,1.002669),
xlim=c(-0.05, 0.05),ylim=c(0,70),type="l",
ylab=NULL, axes=FALSE, col=2)

#PDF,GBP/JPY,alphahat=0.8419174,
sigmahatgbp=0.004655019,
kappa2hat=0.9987206
ff2<-function(x,a,s,k){
 x<-seq(-0.04,0.04, 0.001)
```

```
y<-rep(1, times=length(x))
for(i in 1:length(x)){
if (x[i]>=0)
{y[i]<- ((1/s^a)*(a*k))/(1+k^2)*(x[i]*k)^(a-1)
*exp(-(k*x[i]/s)^a)}
else
{y[i]<-((1/s^a)*(a*k))/(1+k^2)*(-x[i]/k)^(a-1)
*exp(-(-x[i]/(k*s))^a)}
y[i]
}
y
}
x<-seq(-0.04,0.04,by=0.001)
hist(logratiogbp, breaks=20,probability="TRUE",
xlim=c(-0.05, 0.05),
ylim=c(0,60), xlab=NULL, ylab=NULL)
par(new=T)
plot(x,ff2(x,0.8419174,0.004655019,0.9987206),
 xlim=c(-0.05, 0.05),
 ylim=c(0,70),type="l",ylab=NULL,axes=FALSE,col=2)

#QQ plots USD
rho<-rep(0,times=length(logratiousd))
rhofunction<-function(n)
{
for(i in 1 : n)
{
rho[i]<-i/(n+(1/4))
}
return(rho)}
rho<-rhofunction(length(logratiousd))
Xrho1<-function(rho, alphahat, sigmahatusd, kappa1hat)
{ Xrho1<-rep(0, times=length(logratiousd))
for(i in 1:length(logratiousd))
{
if ((kappa1hat^2)/(1+kappa1hat^2)<=rho[i])
Xrho1[i]<-sigmahatusd/kappa1hat*
(log(1/(( 1+kappa1hat^2)*(1-rho[i]))))^(1/alphahat)
else
Xrho1[i]<-(-sigmahatusd)*kappa1hat*(log(kappa1hat^2)
/(( 1+kappa1hat^2)*rho[i])))^(1/alphahat)
}
return(Xrho1)
}
Xrho1(rho, 0.8419174,0.002677868,1.002669)
srho<-sort(rho)
```

```
qf<-Xrho1(srho,0.8419174,0.002677868,1.002669)
sqf<-sort(qf)
sqf
plot(sort(logratiousd),sqf)
# QQ PLOTS GBP
rho<-rep(0,times=length(logratiogbp))
rhofunction<-function(n)
{
for(i in 1 : n)
{
rho[i]<-i/(n+(1/4))
}
return(rho)}
rho<-rhofunction(length(logratiogbp))
Xrho1<-function(rho,alphahat,sigmahatgbp,kappa2hat)
{ Xrho1<-rep(0, times=length(logratiogbp))
for(i in 1:length(logratiogbp))
{
if ((kappa2hat^2)/(1+kappa2hat^2)<=rho[i])
Xrho1[i]<-sigmahatgbp/kappa2hat *(log(1/((1+kappa2hat^2)
*(1-rho[i]))))^(1/alphahat)
else
Xrho1[i]<-(-sigmahatgbp)*kappa2hat*(log(kappa2hat^2)
/(( 1+kappa2hat^2)*rho[i])))^(1/alphahat)
}
return(Xrho1)
}
Xrho1(rho,0.8419174,0.004655019,0.9987206)
srho<-sort(rho)
qf1<-Xrho1(srho,0.8419174,0.004655019,0.9987206)
sqf1<-sort(qf1)
sqf1
plot(sort(logratiogbp),sqf1)

#QQ plots comparison with normal distribution
plot(sort(logratiousd), qnorm(rho, m1hat,sqrt(sigma11hat)),
xlab="logratiousd", ylab="quantiles from normal distribution")
plot(sort(logratiogbp), qnorm(rho, m2hat,sqrt(sigma22hat)),
xlab="logratiogbp", ylab="quantiles from normal)

#HISTOGRAM and PDF,USD/JPY NORMAL DISTRIBUTION
x<-seq(-4,4, 0.0001)
y<-rep(1, times=length(x))
hist(logratiousd,probability="TRUE",xlim=c(-0.05, 0.05),
ylim=c(0,78),xlab=NULL,ylab=NULL)
par(new=T)
```

```
plot(x, dnorm(x,m1hat, sqrt(sigma11hat)),xlim=c(-0.05, 0.05),
ylim=c(0,70),type="l",ylab=NULL,axes=FALSE,col=2)

#HISTOGRAM and PDF,GBP/JPY alpha=0.09,sigma=0.005,
kappa2=0.9981285
x<-seq(-0.04,0.04, 0.001)
y<-rep(1, times=length(x))
hist(logratiogbp,breaks=20,probability="TRUE",
xlim=c(-0.05,0.05),ylim=c(0,60),xlab=NULL,ylab=NULL)
par(new=T)
plot(x, dnorm(x,m2hat,sqrt(sigma22hat)),xlim=c(-0.05, 0.05),
ylim=c(0,70),
type="l",ylab=NULL,axes=FALSE,col=2)

#PROJECTIONS in the direction of the various vectors
a1=c( sqrt(2)/2,sqrt(2)/2)
projmat1<-as.matrix(cbind(logratiousd,logratiogbp))
project1<-projmat1%*%a1
project1mean<-mean(project1)
project1meansquare<-mean(project1^2)
alphahat<- 0.8419174
m1project1<-project1mean/gamma(1+1/alphahat)
sigma11project1<-2*( project1meansquare -(G(alphahat)
*project1mean^2))/f2(alphahat)
sigmaproject1<-sqrt(sigma11project1/2)
kappaproject1<-(sqrt(4*sigma11project1
+m1project1^2)-m1project1)/(2*sqrt(sigma11project1))

#QQ plot
rho<-rep(0, times=length(project1))
rhofunction<-function(n)
{
for(i in 1:n)
{
rho[i]<-i/(n+(1/4))
}
return(rho)}
rho<-rhofunction(length(project1))
Xrho1<-function(rho, alphahat, sigmaproject1,kappaproject1)
{ Xrho1<-rep(0, times=length(project1))
for(i in 1:length(project1))
{
if ((kappaproject1^2)/(1+kappaproject1^2)<=rho[i])
Xrho1[i]<-sigmaproject1/kappaproject1
*(log(1/(( 1+kappaproject1^2)*(1-rho[i]))))^(1/alphahat)
else
```

```
Xrho1[i]<-(-sigmaproject1)*kappaproject1*
(log(kappaproject1^2/(( 1+kappaproject1^2)
*rho[i])))^(1/alphahat)
}
return(Xrho1)
}
Xrho1(rho,0.8419174,0.006,1.00148)
srhoproject1<-sort(rho)
qfproject1<-Xrho1(srho,0.8419174,0.006,1.00148)
sqfproject1<-sort(qfproject1)
plot(sort(project1), sqfproject1,
 xlab="projection with the vector (\sqrt(2)/2,\sqrt(2)/2)",
ylab= "quantiles from Weibull distribution")

a2=c(-sqrt(2)/2,sqrt(2)/2)
projmat2<-as.matrix(cbind(logratiousd,logratiogbp))
project2<-projmat2%*%a2
project2mean<-mean(project2)
project2meansquare<-mean(project2^2)
m1project2<-project2mean/gamma(1+1/alphahat)
sigma11project2<-2*(project2meansquare
-(G(alphahat)*project2mean^2))/f2(alphahat)
sigmaproject2<-sqrt(sigma11project2/2)
kappaproject2<-(sqrt(4*sigma11project2+m1project2^2)-
m1project2)/(2*sqrt(sigma11project2))

#QQ plot
rho<-rep(0, times=length(project2))
rhofunction<-function(n)
{
for(i in 1 : n)
{
rho[i]<-i/(n+(1/4))
}
return(rho)}
rho<-rhofunction(length(project2))
Xrho1<-function(rho, alphahat, sigmaproject2,kappaproject2)
{Xrho1<-rep(0, times=length(project2))
for(i in 1:length(project2))
{
if ((kappaproject2^2)/(1+kappaproject2^2)<=rho[i])
Xrho1[i]<-sigmaproject2/kappaproject2*
(log(1/(( 1+kappaproject2^2)*(1-rho[i]))))^(1/alphahat)
else
Xrho1[i]<-(-sigmaproject2)*kappaproject2
*(log(kappaproject2^2)
```

```
/(( 1+kappaproject2^2)*rho[i])))^(1/alphahat)
}
return(Xrho1)
}
Xrho1(rho,0.8419174,0.006,1.00148)
srhoproject2<-sort(rho)
qfproject2<-Xrho1(srho,0.8419174,0.006,1.00148)
sqfproject2<-sort(qfproject2)
plot(sort(project2),sqfproject2,
xlab="projection with the vector (-\sqrt(2)/2,\sqrt(2)/2)",
ylab="quantiles from Weibull distribution")

a3=c(sqrt(3),  -sqrt(3))
projmat3<-as.matrix(cbind(logratiousd,logratiogbp))
project3<-projmat3%*%a3
project3mean<-mean(project3)
project3meansquare<-mean(project3^2)
m1project3<-project3mean/gamma(1+1/alphahat)
sigma11project3<-2*( project3meansquare -
(G(alphahat)*project3mean^2))
/f2(alphahat)
sigmaproject3<-sqrt(sigma11project3/2)
kappaproject3<-(sqrt(4*sigma11project3+m1project3^2)-m1project3)/
(2*sqrt(sigma11project3))

#QQ plots
rho<-rep(0, times=length(project3))
rhofunction<-function(n)
{
for(i in 1:n)
{
rho[i]<-i/(n+(1/4))
}
return(rho)}
rho<-rhofunction(length(project3))
Xrho1<-function(rho,alphahat,sigmaproject3,kappaproject3)
{Xrho1<-rep(0, times=length(project3))
for(i in 1:length(project3))
{
if ((kappaproject3^2)/(1+kappaproject3^2)<=rho[i])
Xrho1[i]<-sigmaproject3/kappaproject3*
(log(1/((1+kappaproject3^2)*(1-rho[i]))))^(1/alpha)
else
Xrho1[i]<-(-sigmaproject3)*kappaproject3
*(log(kappaproject3^2)/(( 1+kappaproject3^2)*rho[i])))^(1/alpha)
}
```

```
return(Xrho1)
}
Xrho1(rho,0.8419174,0.006546593,1.002806)
srhoproject3<- sort(rho)
qfproject3<-Xrho1(srho,0.8419174,0.006546388,1.002806)
sqfproject3<-sort(qfproject3)
plot(sort(project3),sqfproject3,
xlab="projection with the vector (\sqrt(3),-\sqrt(3))",
ylab="quantiles from Weibull distribution")

a4=c(-sqrt(3),-sqrt(3))
projmat4<-as.matrix(cbind(logratiousd,logratiogbp))
project4<-projmat4%*%a4
project4mean<-mean(project4)
project4meansquare<-mean(project4^2)
m1project4<-project4mean/gamma(1+1/alphahat)
sigma11project4<-2*(project4meansquare
-(G(alphahat)*project4mean^2))/f2(alphahat)
sigmaproject4<-sqrt(sigma11project4/2)
kappaproject4<-(sqrt(4*sigma11project4+m1project4^2)
-m1project4)/(2*sqrt(sigma11project4))

#QQ plot
rho<-rep(0, times=length(project4))
rhofunction<-function(n)
{
for(i in 1:n)
{
rho[i]<-i/(n+(1/4))
}
return(rho)}
rho<-rhofunction(length(project4))
Xrho1<-function(rho,alphahat,sigmaproject4,kappaproject4)
{ Xrho1<-rep(0, times=length(project4))
for(i in 1:length(project4))
{
if ((kappaproject4^2)/(1+kappaproject4^2)<=rho[i])
Xrho1[i]<-sigmaproject4/kappaproject4
*(log(1/(( 1+kappaproject4^2)*(1-rho[i]))))^(1/alphahat)
else
Xrho1[i]<-(-sigmaproject4)*kappaproject4*(log(kappaproject4^2)
/(( 1+kappaproject4^2)*rho[i])))^(1/alphahat)
}
return(Xrho1)
}
Xrho1(rho, 0.8419174,0.01435818,0.9998524)
```

```
srhoproject4<- sort(rho)
qfproject4<-Xrho1(srho,0.8419174,0.01435818,0.9998524)
sqfproject4<-sort(qfproject4)
plot(sort(project4),sqfproject4,
xlab="projection with the vector (-\sqrt(3),-\sqrt(3))",
ylab="quantiles from Weibull distribution" )

#Plotting
par(mfrow=c(3,2))
plot(sort(project1), sqfproject1,
xlab="projection with the vector (\sqrt(2)/2,\sqrt(2)/2)",
ylab="quantiles from Weibull distribution")
plot(sort(project1), qnorm(rho, project1mean,sqrt(sigma11project1)),
xlab="projection with the vector (\sqrt(2)/2,\sqrt(2)/2)",
ylab="quantiles from Normal distribution")
plot(sort(project2),sqfproject2,
xlab="projection with the vector(-\sqrt(2)/2,\sqrt(2)/2)",
ylab="quantiles from Weibull distribution")
plot(sort(project2), qnorm(rho, project2mean,sqrt(sigma11project2)),
xlab="projection with the vector(-\sqrt(2)/2,\sqrt(2)/2)",
ylab="quantiles from Normal distribution")
plot(sort(project3),sqfproject3,
xlab="projection with the vector (\sqrt(3),-\sqrt(3))",
ylab="quantiles from Weibull distribution" )
plot(sort(project3), qnorm(rho, project3mean ,sqrt(sigma11project3)),
xlab="projection with the vector (\sart(3),-\sqrt(3))",
ylab="quantiles from Normal distribution")

#SCATTERPLOT
projmat<-as.matrix(cbind(logratiousd,logratiogbp))
plot(projmat,main ="Scatterplot logratio USD vs. logratio GBP",
xlab="logratio USD",ylab="logratio GBP")

#3DHISTOGRAM

#Create cuts:
 x_c2 <- cut(logratiousd, 40)
 y_c2 <- cut(logratiogbp, 40)
 z2 <- table(x_c2, y_c2)
 hist3D(z=z2, border="black")
```

APPENDIX B: Daljši povzetek disertacije v slovenskem jeziku

Asimetrična večrazsežna Weibullova porazdelitev

Kratek povzetek

Disertacija obravnava nesimetrične Weibullove porazdelitve tako v eni kot v več dimenzijah. V disertaciji so predstavljene deloma nove definicije teh porazdelitev in izpeljane metode za ocenjevanje parametrov, kar je nujna predpostavka za uporabo pri modeliranju finančnih podatkov kot so donosi finančnih naložb ali modeliranje menjalnih tečajev. Po obširnem pregledu reyultatov v eni dimenziji, je predstavljena posplošitev na več dimenzij. Ta posplošitev je dejanski prispevek disertacije. Posplošitev je posredna prek reprezentacij nesimetrične Laplacove porazdelitve v eni dimenziji. Navedene so lastnosti te nove družine porazdelitev in obravnavana vprašanja ocenjevanja parametrov in simulacij. Na koncu so predstavljene uporabe te nove družine porazdelitev na dejanskih menjalnih tečajih.

UVOD

Weibullova porazdelitev je ena od standardnih porazdelitev v statistiki. Dobimo jo tako, da eksponentno slučajno spremenljivko potenciramo na potenco $1/\alpha$ in dobimo slučajno spremenljivko z gostoto

$$f(x) = \alpha x^{\alpha-1} e^{-x^\alpha}, \ \ x > 0\,. \tag{271}$$

Porazdelitev je poimenovana po Waloddiju Weibullu, ki jo je uporabljal za modeliranje nateznih trdnosti materijalov (glej Weibull [96, 97]), kontroli kvalitete in teoriji zanesljivosti (glej [98]). Ta porazdelitev je eden od najpogostejših statističnih modelov (glej Halinan [28]) kot tudi obširno bibliografijo v Johnson at al. (glej [40]).

Weibullovo porazdelitev lahko razširimo na celotno realno os s simetrizacijo gostote (271), kar pomeni, da je gostota enaka

$$f(x) = \frac{\alpha}{2}|x|^{\alpha-1} e^{-|x|^\alpha}, \ \ x \neq 0. \tag{272}$$

Simetrične univariatne Weibullove porazdelitve so našle številne uporabe tako kot modeli donosov finančih naložb kot tudi na drugih področjih, recimo v zavarovalništvu kot model za porazdelitve izgub (glej Chenyao et al. [9], Hürlimann [34], Mittnik and Rachev [74]).

Obstaja obširno empirično dokazno gradivo, da logaritmi donosov finančnih naložb niso simetrično porazdeljeni. Zato je pri modeliranju takšnih finančnih podatkov potrebno to dejstvo upoštevati in iskati družine porazdelitev, ki niso simetrične. Simetrično

Weibullovo porazdelitev lahko na več načinov posplošimo tako, da postane asimetrična (272). Fernandez and Steel (glej [19]) uvedeta inverzna faktorja, s katerima simetrično porazdelitev pretvorita v asimetrično, tako da je nova gostota enaka

$$g(x) = \frac{\kappa}{\sigma(1+\kappa^2)} \begin{cases} f(x\kappa/\sigma), & x > 0 \\ f(\frac{x}{\sigma\kappa}), & x < 0, \end{cases} \tag{273}$$

kjer je $\kappa > 0$. S tem dobimo asimetrično Weibullovo porazdelitev s parametri α, σ in κ. V enodimenzionalnem primeru dobimo tako vsestransko uporabno družino porazdelitev, ki omogoča modeliranje in ocenjevanje parametrov. Vendar ta pristop ne omogoča posplošitev na večrazsežne porazdelitve.

Posplošitev na večrazsežni primer poteka s pomočjo ustrezne reprezentacije enodimenzionalne asimetrične Weibullove porazdelitve. Če ima Y standardno Laplacovo porazdelitev, potem velja enakost v porazdelitvi

$$Y \stackrel{d}{=} \sqrt{2E}Z, \tag{274}$$

kjer je E standardna eksponentna slučajna spremenljivka neodvisna od standardizirano normalne slučajne spremenljivke Z, (glej Kotz et al. [49]). Gre torej za mešanico normalnih porazdelitev s slučajno varianco. Kozubowski and Podgórski (glej [48]) sta pokazala, da za slučajno spremenljivko Y z asimetrično Laplaceovo porazdelitvijo po definiciji (273) velja

$$Y \stackrel{d}{=} mE + \sqrt{2E}Z\,. \tag{275}$$

Po drugi strani za simetrično Laplaceovo slučajno spremenljivko L in od nje neodvisno stabilno slučajno spremenljivko S z indeksom $\alpha \in (0, 1]$ definirano z Laplaceovo transformacijo

$$g(t) = \mathbb{E}e^{-tS} = \int_0^\infty e^{-st} f_\alpha(s)ds = e^{-t^\alpha}\,, \tag{276}$$

velja, da je ima spremenljivka $Y = L/S$ simetrično Weibullovo porazdelitev s parametroma α in $\sigma = 1$. To dejstvo in enakost v (275) vodita do naslednje reprezentacije asimetrične Weibullove porazdelitve:

$$W \stackrel{d}{=} \frac{mE + \sqrt{2E}X}{S} \tag{277}$$

pri čemer so E, $X \sim \mathrm{N}(0, \tau^2)$ in S neodvisne. Elementaren račun pokaže, da ima slučajna spremenljivka v (277) asimetrično Weibullovo porazdelitev v smislu (273) s parametri α,

$$\sigma = \tau \qquad \text{in} \qquad \kappa = \frac{\sqrt{m^2 + 4\tau^2} - m}{2\tau}\,.$$

Reprezentacijo (277) lahko naravno posplošimo na več dimenzij, s tem da slučajno spremenljivko X zamenjamo z multivariatno normalnim vektorjem $\mathbf{X}$. S tem se zavestno odpovemo možnosti, da bi bile komponente neodvisne, vendar ima ta nova družina porazdelitev številne ugodne lastnosti. Ta pot je lahko alternativa kopulam, ki se pogosto uporabljajo za konstrukcijo porazdelitev z danimi robnimi porazdelitvami. Definiramo

$$\mathbf{W} = \frac{\mathbf{m}E + \sqrt{2E}\mathbf{X}}{S} \tag{278}$$

kjer je $\mathbf{m} \in \mathbb{R}^d$, $d \times d$ simetrična matrika $\boldsymbol{\Sigma}$ pa je pozitivno semi-definitna. Oznaka $\mathbf{X} \sim \mathrm{N}_d(\mathbf{0}, \boldsymbol{\Sigma})$ pomeni, da ima $\mathbf{X}$ večrazsežno normalno porazdelitev s pričakovano vrednostjo $\mathbf{m}$ in kovariančno matriko $\boldsymbol{\Sigma}$. Iz reprezentacije (277) sledi, da so robne porazdelitve vse asimetrične Weibullove v smislu definicije (273), tako da je da je poimenovanje *asimetrična večrazsežna Weibullova porazdelitev* naravno.

Zgoraj definirano družino porazdelitev bi lahko uporabili za modeliranje multivariatnih finančnih podatkov. Posplošitev res ne dopušča neodvisnih komponent, vendar od normalne porazdelitve podeduje številne lastnosti kot recimo to, da so vse robne porazdelitve istega tipa in vse linearne kombinacije komponent asimetrične Weibullove. Pri pogojnih porazdelitvah se lastnosti normalnih vektorjev ne prenesejo v lepi obliki. Omejiti se moramo tudi na parametre $\alpha \in (0, 1]$, vendar se izkaže, da samo v tem primeru dobimo unimodalne porazdelitve, ki so vsebinsko primerne za modeliranje.

ASIMETRIČNA VEČRAZSEŽNA WEIBULLOVA PORAZDELITEV

Izpeljave v prejšnjem razdelku vodijo do naslednje matematične definicije:
Definicija 1: *Slučajni vektor* $\mathbf{W} \in \mathbb{R}^d$ *ima asimetrično večrazsežno Weibullovo porazdelitev s parametri* $0 < \alpha \leq 1$, $\mathbf{m} \in \mathbb{R}^d$*in* $\boldsymbol{\Sigma}$, *ki jo označimo z* $\mathrm{W}_d(\alpha, \mathbf{m}, \boldsymbol{\Sigma})$, *če lahko zapišemo*

$$\mathbf{W} \stackrel{d}{=} \frac{\mathbf{m}E + \sqrt{2E}\mathbf{X}}{S}. \tag{279}$$

Pri tem je E *standardna eksponentna slučajna spremenljivka,* S *od nje neodvisna stabilna slučajna spremenljivka dana z Laplacovo transformacijo*

$$\mathbb{E}e^{-tS} = e^{-t^\alpha}, \tag{280}$$

in $\mathbf{X} \in \mathbb{R}^d$ *večrazsežen normalni vektor s pričkovano vrednostjo* 0 *in kovariančno matriko* $\boldsymbol{\Sigma}$, *torej* $\mathbf{X} \sim \mathrm{N}_d(0, \boldsymbol{\Sigma})$, *neodvisen od* (E, S).

Opomba 1. Za $m = 0$ so vse robne porazdelitve simetrične.
Opomba 2 Za $\alpha = 1$ je $S = 1$ in dobimo asimetrične Laplacove porazdelitve kot mejni primer.
Opomba 3 Za $\alpha = \frac{1}{2}$ je gostota slučajne spremenljivke S znana Lévyjeva gostota dana

z

$$f_S(s) = \frac{1}{2\sqrt{\pi s^3}} e^{-\frac{1}{4s}} \ , s > 0. \tag{281}$$

OPOMBA 4 Za obrnljivo matriko $\mathbf{\Sigma}$ ima večrazsežna asimetrična Weibullova porazdelitev gostoto. Iz definicij sledi

$$(\mathbf{W}|E = e, S = s) \sim \mathrm{N}_d\left(\frac{\mathbf{m}e}{s}, \frac{2e}{s^2}\mathbf{\Sigma}\right). \tag{282}$$

Iz tega lahko načeloma dobimo gostoto z integracijo, za primer $d = 3$ pa je možno gostoto izračunati v ekplicitni obliki (glej razdelek 6).

POLARNA REPREZENTACIJA

Kot prvo bomo pokažemo, da družina porazdelitev iz Definicije 1 spada med eliptične porazdelitve. Po definiciji gre za porazdelitve, ki imajo za matriko $\mathbf{\Sigma}$ gostoto oblike

$$f(x) = k_d|\mathbf{\Sigma}|^{-\frac{1}{2}} g[(\mathbf{x} - \mathbf{m})'\mathbf{\Sigma}^{-1}(\mathbf{x} - \mathbf{m})], \tag{283}$$

za funkcijo g ene spremenljivke, k_d pa je ustrezna normalizacijska konstanta. Razred takih porazdelitev označimo z $EC_d(\mathbf{m}, \mathbf{\Sigma}, g)$. Vsako porazdelitev slučajnega vektorja $\mathbf{Y}$ s porazdelitvijo iz $\mathbf{Y} \sim EC_d(\mathbf{0}, \mathbf{\Sigma}, g)$ lahko predstavimo kot

$$\mathbf{Y} \stackrel{d}{=} R\mathbf{H}\mathbf{U}^{(d)} \tag{284}$$

kjer je $\mathbf{H}$ taka $d \times d$ matrika, da je $\mathbf{H}\mathbf{H}' = \mathbf{\Sigma}$, R je pozitivna slučajna spremenljivka in $\mathbf{U}^{(d)}$ slučajni vektor enakomerno porazdeljen po površini enotske krogle v $\mathbb{R}^d$ neodvisen od R, (glej Fang, et al. [17]). Konkretno ima R porazdelitev enako porazdelitvi $\sqrt{\mathbf{Y}'\mathbf{\Sigma}^{-1}\mathbf{Y}}$, medtem ko je $\mathbf{H}\mathbf{U}^{(d)}$ enakomerno porazdeljena po površini elipsoida $\{\mathbf{y} \in \mathbb{R}^d : \mathbf{y}\mathbf{\Sigma}^{-1}\mathbf{y} = 1\}$, (glej Kotz et al. [48]).

Trditev 1: *Naj bo* $\mathbf{Y} \sim W_d(\alpha, 0, \mathbf{\Sigma})$*, kjer predpostavljamo* $|\mathbf{\Sigma}| > 0$*. Slučajni vektor* $\mathbf{Y}$ *dopušča polarno reprezentacijo oblike (284), kjer je* $\mathbf{H}$ *taka* $d \times d$ *matrika, da je* $\mathbf{H}\mathbf{H}' = \mathbf{\Sigma}$*,* $\mathbf{U}^{(d)}$ *je enakomerno porazdeljen vektor na površini enotske krogle v* $\mathbb{R}^d$*, neodvisen od slučajne spremenljivke* R *z gostoto*

$$f_R(z) = \frac{\sqrt{2}d}{2^{\frac{d}{2}}\Gamma\left(\frac{d}{2}+1\right)} \int_0^\infty \left(\frac{z}{y}\right)^{d-1} e^{-\frac{1}{2}\left(\frac{z}{y}\right)^2} \int_0^\infty \sqrt{x} e^{-y^2 x} f_\alpha(\sqrt{x}) dx dy. \tag{285}$$

Dokaz: Po Definiciji 1 ima vektor $\mathbf{Y}$ reprezentacijo (279). Naj bo $\mathbf{m} = 0$ in $\mathbf{\Sigma} = \mathbf{H}\mathbf{H}'$, kjer je $\mathbf{H}$ spodnja trikotna matrika. Slučajni vektor $\mathbf{X} \sim \mathrm{N}_d(0, \mathbf{\Sigma})$ ima reprezentacijo $\mathbf{X} = \mathbf{H}\mathbf{N}$, kjer je $\mathbf{N} \sim \mathrm{N}_d(0, \mathbf{I})$. Znano je, da ima $\mathbf{N}$ reprezentacijo $\mathbf{N} \stackrel{d}{=} R_N\mathbf{U}^d$, kjer sta R_N in $\mathbf{U}^{(d)}$ neodvisna, $\mathbf{U}^{(d)}$ je enakomerno porazdeljen po površini enotske krogle v $\mathbb{R}^d$ in je $R_N^2 \sim \chi^2(d)$. Sklepamo lahko, da je dovolj dokazati, da ima $\sqrt{\frac{2E}{S^2}}R_N$ gostoto (285), kar dokazuje trditev, slednje pa sledi z neposrednim preverjanjem.

LINEARNE TRANSFORMACIJE

Večrazsežna asimetrična Weibullova porazdelitev podeduje nekatere lastnosti večrazsežne normalne porazdelitve.

Trditev 2: *Naj bo* $\mathbf{W} = (W_1, W_2, ..., W_d)' \sim W_d(\alpha, \mathbf{m}, \mathbf{\Sigma})$ *in naj bo* $\mathbf{A}$ *realna* $l \times d$ *matrika. Slučajni vektor* $\mathbf{AW}$ *je porazdeljen kot* $W_l(\alpha, \mathbf{m}_A, \mathbf{\Sigma}_A)$*, kjer je* $\mathbf{m_A} = \mathbf{Am}$ *in* $\mathbf{\Sigma_A} = \mathbf{A\Sigma A}'$.

Dokaz: Opazimo, da je

$$\mathbf{AW} = \frac{\mathbf{A}(\mathbf{m}E + \sqrt{2E}\mathbf{X})}{S} = \frac{\mathbf{Am}E + \sqrt{2E}\mathbf{AX}}{S} \tag{286}$$

kjer ima $\mathbf{AW}$ pričakovano vrednost $\mathbf{m_A} = \mathbf{Am}$ in je $\mathbf{AX} \sim \mathrm{N}_l(\mathbf{0}, \mathbf{A\Sigma A}')$. Iz zgornjega sledi, da so vse robne porazdelitve asimetrične Weibullove porazdelitve iz družine asimetričnih Weibullovih porazdelitev. Dejstvo, da lahko najdemo večrazsežno porazdelitev s asimetričnimi Weibullovimi robnimi porazdelitvami, pri katerih so vse linearne kombinacije spet asimetrične Weibullove, je presenetljivo, saj to za neodvisne slučajne spremenljivke z asimetrično Weibullovo porazdelitvijo ne velja.

Korolar 1: *Naj bo* $\mathbf{W} = (W_1, W_2, ..., W_d)' \sim W_d(\alpha, \mathbf{m}, \mathbf{\Sigma})$*, kjer je* $\mathbf{\Sigma} = (\sigma_{i,j})_{i,j=1}^d$. *Velja,*

(i) Za vse $k \leq d$ *je* $(W_1, ..., W_k) \sim W_k(\alpha, m', \Sigma')$*, kjer je* $m' = (m_1, ..., m_k)'$ *in je* Σ' $k \times k$ *matrika s* $\sigma'_{i,j} = \sigma_{i,j}$ *za* $i, j = 1, ...k$*;*

(ii) Za katerikoli vektor $\mathbf{b} = (b_1, ..., b_d)' \in \mathbf{R}^d$ *ima slučajna spremenljivka* $W_{\mathbf{b}} = \sum_{k=1}^d b_k W_k$ *enorazsežno asimetrično Weibullovo porazdelitev s parametri* $\sigma = \sqrt{\mathbf{b}'\mathbf{\Sigma b}}$ *in* $\mu = \mathbf{m}'\mathbf{b}$.

(iii) Vse robne porazdelitve so asimetrične Weibullove.

Dokaz: Za (i) lahko uporabimo Trditev 2 s $k \times d$ matriko $A = (a_{i,j})$ tako da je $a_{i,i} = 1$ in $a_{i,j} = 0$ za $i \neq j$. Za (ii) uporabimo Trditev 2 z $l = 1$. Za (iii) uporabimo definicijo $W_d(\alpha, \mathbf{m}, \mathbf{\Sigma})$ porazdelitve in dejstvo, da so robne porazdelitve večrazsežne normalne porazdelitve večrazsežne normalne.

POGOJNE PORAZDELITVE

Naj bo $\mathbf{W} \sim W_d(\alpha, \mathbf{m}, \mathbf{\Sigma})$. Poznavanje pogojnih porazdelitev je pogosto pomembno, vendar se v primeru večrazsežnih asimetričnih Weibullovih porazdelitev lastnosti večrazsežne normalne porazdelitve ne prenesejo na lep način.

Naj bo

$$\mathbf{W} = \begin{pmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \end{pmatrix} \sim W_d(\alpha, \mathbf{m}, \mathbf{\Sigma})$$

s pripadajočima dimenzijama d_1 in d_2, $d_1 + d_2 = d$ ter

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \sim \mathrm{N}_d(\mathbf{0}, \mathbf{\Sigma}),$$

z

$$\mathbf{m} = \begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix} \qquad \text{in} \quad \mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{pmatrix}$$

Z zgornjim oznakami velja

Trditev 3: *Privzemite, da je* $|\mathbf{\Sigma}_{22}| > 0$ *in* $\mathbf{m}_2 = 0$. *Porazdelitev vektorja*

$$\mathbf{W}_1 - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{W}_2 \tag{287}$$

je $W_{d_1}\left(\alpha, \mathbf{m}_1, \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}\right)$.

Dokaz: Vektor $\mathbf{X}_1 - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2$ je neodvisen od $(E, S, \mathbf{W}_2)$. Iz tega sledi, da je

$$\frac{\mathbf{m}_1 E + \sqrt{2E}\left(\mathbf{X}_1 - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2\right)}{S} \sim W_{d_1}\left(\alpha, \mathbf{m}_1, \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}\right) .$$

Vektor v (287) ni neodvisen od $\mathbf{W}_2$ kot pri večrazsežni normalni porazdelitvi, zato analogij ni mogoče razširiti na asimetrično Weibullovo porazdelitev. V nekaterih primerih je možno izračunati pogojno pričakovano vrednost in pogojno varianco.

PRIČAKOVANE VREDNOSTI IN KOVARIANCE

Predstavitev nove družine porazdelitev vključuje tudi izračun momentov. Za ta namen potrebujemo momente stabilne slučajne spremenljivke S. Po eni strani je

$$\int_0^\infty t^{\beta-1}\mathbb{E}(e^{-tS})dt = \int_0^\infty t^{\beta-1}e^{-t^\alpha}dt = \frac{1}{\alpha}\Gamma\left(\frac{\beta}{\alpha}\right) , \tag{288}$$

po drugi strani pa

$$\int_0^\infty t^{\beta-1}\mathbb{E}(e^{-tS})dt = \mathbb{E}\left(\int_0^\infty t^{\beta-1}e^{-tS}dt\right) = \mathbb{E}\left(\frac{\Gamma(\beta)}{S^\beta}\right) . \tag{289}$$

Sledi, da je

$$\Gamma(\beta)\mathbb{E}[S^{-\beta}] = \frac{1}{\alpha}\Gamma\left(\frac{\beta}{\alpha}\right) = \frac{1}{\beta}\Gamma\left(\frac{\beta}{\alpha} + 1\right). \tag{290}$$

Za poseben primer $\beta = n$ dobimo

$$\mathbb{E}[S^{-n}] = \frac{1}{n}\frac{\Gamma(\frac{n}{\alpha}+1)}{\Gamma(n)} = \frac{\Gamma\left(\frac{n}{\alpha}+1\right)}{\Gamma(n+1)} \tag{291}$$

Računamo lahko

$$\mathbb{E}\left(\mathbf{W}\right) = \mathbb{E}\left(\mathbb{E}\left(\mathbf{W}|E,S\right)\right) = \mathbf{m}\Gamma\left(\frac{1}{\alpha}+1\right) \tag{292}$$

in posledično

$$\begin{aligned} \mathrm{cov}(W_i,W_j) &= \mathbb{E}\left(\mathrm{cov}\left(W_i,W_j|E,S\right)\right) + \mathrm{cov}\left(\mathbb{E}\left(W_i|E,S\right),\mathbb{E}\left(W_j|E,S\right)\right) \quad (293)\\ &= \sigma_{ij}\mathbb{E}\left(\frac{2E}{S^2}\right) + \mathrm{cov}\left(\frac{m_iE}{S},\frac{m_jE}{S}\right)\\ &= \sigma_{ij}\Gamma\left(\frac{2}{\alpha}+1\right) + m_im_j\left[\Gamma\left(\frac{2}{\alpha}+1\right) - \Gamma\left(\frac{1}{\alpha}+1\right)^2\right]. \end{aligned}$$

GOSTOTE IN SIMULACIJE

Uporabe asimetrične Weibullove porazdelitve so odvisne od učinkovitih metod za ocenjevanje parametrov. Metoda največjega verjetja zahteva poznavanje gostot. Za stabilne slučajne spremenljivke so eksplicitne gostote znane le v nekaterih primerih, recimo ko je $\alpha = 1/2$. Privzemimo, da je matrika $\mathbf{\Sigma}$ obrnljiva. Iz Definicije 1 in lastnosti večrazsežne normalne porazdelitve sledi, da je

$$f_{\mathbf{W}|E=u,S=s}(\mathbf{w}) = \frac{s^d}{(2\pi)^{d/2}(2u)^{d/2}\sqrt{|\mathbf{\Sigma}|}}\exp\left[-\frac{s^2}{4u}\left(\mathbf{w}-\frac{\mathbf{m}u}{s}\right)^T\mathbf{\Sigma}^{-1}\left(\mathbf{w}-\frac{\mathbf{m}u}{s}\right)\right]. \tag{294}$$

Iz pogojne gostote bi načeloma lahko izračunali gostoto, vendar je gostota $f_S(s)$ eksplicitno znana le v nekaterih primerih. Kljub temu je gostote možno izraziti z integrali elementarnih funkcij in jih v primeru, ko je d lih, tudi eksplicitno izračunati. Označimo

$$a = \mathbf{w}'\mathbf{\Sigma}^{-1}\mathbf{w}, \quad b = \mathbf{w}'\mathbf{\Sigma}^{-1}\mathbf{m} \quad \text{in} \quad c = \mathbf{m}'\mathbf{\Sigma}^{-1}\mathbf{m}\,. \tag{295}$$

Prepišemo (202)

$$f_{\mathbf{W}|E=u,S=s}(\mathbf{w}) = \frac{s^d}{(2\pi)^{d/2}(2u)^{d/2}\sqrt{|\mathbf{\Sigma}|}}\exp\left[-\frac{as^2}{4u}+\frac{bs}{2}-\frac{cu}{4}\right], \tag{296}$$

množimo z $f_E(u)f_S(s)$ in integriramo po u. Dobimo gostoto para $(\mathbf{W},S)$. Po formuli (5.34) v [79] za $\mu,p,q>0$ velja

$$\int_0^\infty \frac{1}{u^\mu}e^{-\frac{p}{t}-qt}dt = 2\left(\frac{p}{q}\right)^{-\mu/2+1/2}K_{\mu-1}\left(2\sqrt{pq}\right) \tag{297}$$

kjer je $K_\nu(z)$ modificirana Besslova funkcija z indeksom ν. Označimo $\beta = \sqrt{a\left(\frac{c}{4}+1\right)}$. Z uporabo (297) po nekaj preurejanja sledi

$$f_{\mathbf{W},S}(\mathbf{w},s) = \frac{s^{d/2+1}}{(2\pi)^{d/2}\sqrt{|\mathbf{\Sigma}|}} e^{\frac{bs}{2}} \left(\frac{\beta}{a}\right)^{d/2-1} K_{d/2-1}\left(\beta s\right) f_S(s)\,. \tag{298}$$

Po formuli (3.20) v [79] ima funkcija $K_\nu(p)$ integralsko reprezentacijo

$$K_\nu(p) = \frac{\sqrt{\pi}(p/2)^\nu e^{-p}}{\Gamma(\nu+1/2)} \int_0^\infty e^{-pt}\left[t(t+2)\right]^{\nu-1/2} dt\,. \tag{299}$$

za $p > 0, \nu > -1/2$. Za $y > 0$ označimo

$$\phi_d(y) = E\left(S^d e^{-yS}\right) = (-1)^d \frac{d^n}{dt^n}\left(e^{-t^\alpha}\right)\Big|_{t=y}\,.$$

Funkcija ϕ_d je elementarna, vendar so izrazi za večje d zapleteni. Z uporabo reprezentacije (299) in (298) in z zamenjavo vrstnega reda integracije dobimo

$$\begin{aligned} f_{\mathbf{W}}(\mathbf{w}) &= \\ &= \frac{\sqrt{\pi}\left(\frac{\beta}{a}\right)^{d/2-1}\left(\frac{\beta}{2}\right)^{d/2-1}}{(2\pi)^{d/2}\Gamma\left(\frac{d}{2}-\frac{1}{2}\right)\sqrt{|\mathbf{\Sigma}|}} \int_0^\infty \left[t(t+2)\right]^{d/2-3/2} \phi_d\left(\beta(1+t)-b/2\right) dt. \end{aligned} \tag{300}$$

Po Cauchy-Schwarzovi neenačbi je $\beta \geq b/2$ za $\mathbf{w} \neq 0$, tako da je za $\mathbf{w} \neq 0$ argument v ϕ_d pozitiven. Te oblike gostot dopuščajo implementacijo metode največjega verjetja. Za $d = 2$ in $d = 3$ se izrazi še poenostavijo. Dobimo

$$f_{\mathbf{W}}(\mathbf{w}) = \frac{1}{2\pi\sqrt{|\mathbf{\Sigma}|}} \int_0^\infty \frac{1}{\sqrt{t(t+2)}} \phi_2\left(\beta(1+t) - \frac{b}{2}\right) dt\,. \tag{301}$$

za $d = 2$ in za $d = 3$ z upoštevanjem $K_{1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}$

$$f_{\mathbf{W}}(\mathbf{w}) = \frac{1}{4\pi\sqrt{|\mathbf{\Sigma}|}} a^{-1/2} \phi_2(\beta - b/2) \tag{302}$$

kar je elementaren izraz. Glede na to, da gostote $f_S(s)$ ne poznamo, je dejstvo presenetljivo. Omenimo, da je načeloma za lihe d funkcija $K_{d/2-1}$ elementarna in se gostote načeloma izražajo v sklenjeni obliki.

Kljub temu, da za gostoto v splošnem ne obstajajo ekplicitne formule, pa je možno asimetrične Weibullove porazdelitve učinkovito simulirati. Chambers, Mellow and Stuck (glej [10]) so izpeljali učikovit algoritem, ki ga je kasneje prilagodil Weron (glej [99]), s katerim lahko simuliramo stabilne slučajne spremenljivke s parametrom $\alpha \in (0, 1)$. Postopek je naslednji:

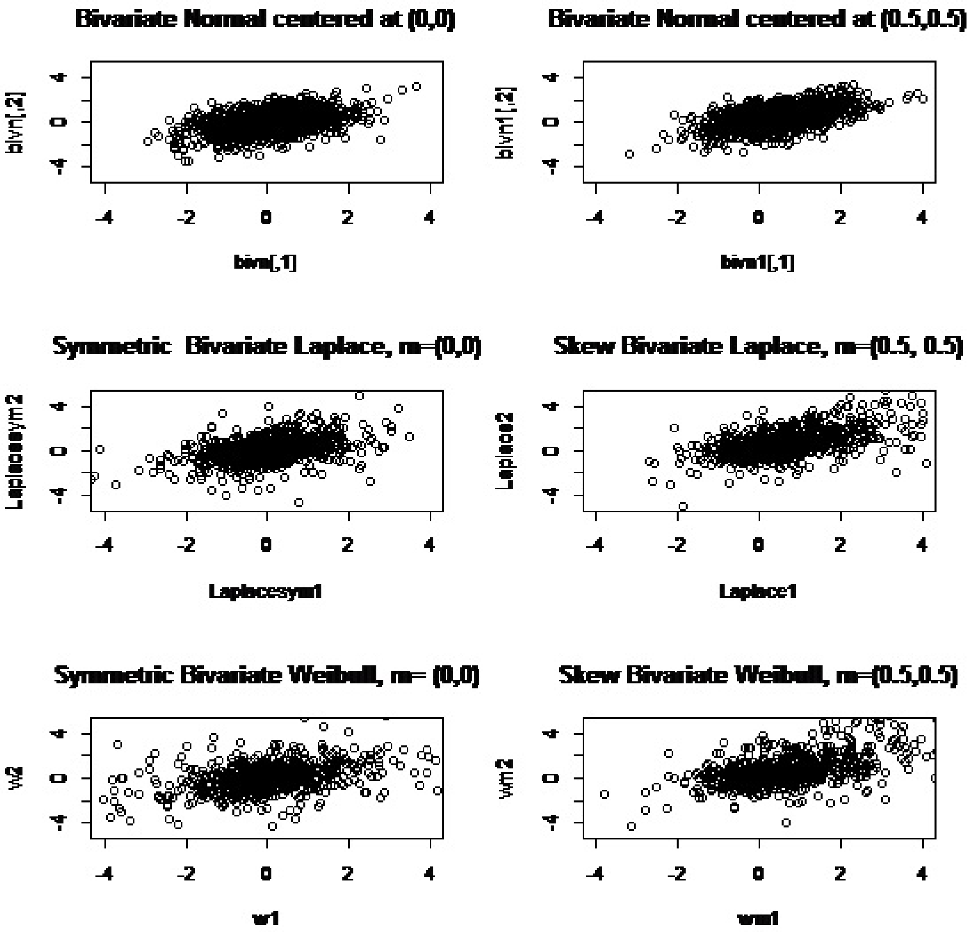

Figure 15: Bivariatne gostote $(n = 1000)$, $\alpha = 0.9$

- Generiramo enakomerno porazdeljeno slučajno spremenljivko $X \sim U(0, \pi)$.
- Na X uporabimo funkcijo

$$U_\alpha(x) = \frac{[\sin(\alpha x)]^{\frac{\alpha}{1-\alpha}} \sin[(1-\alpha)x]}{(\sin x)^{\frac{1}{1-\alpha}}}$$

- Generiramo eksponentno slučajno spremenljivko $E \sim \exp(1)$.
- Izračunamo $S = \left[\frac{U_\alpha(X)}{E}\right]^{\frac{1-\alpha}{\alpha}}$. Slučajna spremenljivka S ima standardno stabilno porazdelitev dano z (280).

Če sledimo definiciji $W_d(\alpha, \mathbf{m}, \mathbf{\Sigma})$ porazdelitve, lahko strnemo algoritem simulacije v naslednje korake:

- Za α generiramo stabilno slučajno spremenljivko S po zgornjem algoritmu.

- Generiramo od S neodvisno standardno eksponentno spremenljivko E.
- Generiramo od S in E neodvisen normalno porazdeljen slučajni vektor $\mathbf{X} \sim \mathrm{N}_d(\mathbf{0}, \mathbf{\Sigma})$.
- Izračunamo $\mathbf{W}$ z uporabo reprezentacije (279).

Slika 1 prikazuje simulirane točke z asimetrično Weibullovo porazdelitvijo za $d = 2$. Iz slik je mogoče razbrati asimetrično naravo porazdelitev in vpliv izbire parametrov.

UPORABE

Enorazsežno asimetrično Weibullovo porazdelitev so številni avtorji uspešno uporabili kot model za finančne podatke. Disertacija predstavi uporabo bivariatne asimetrične Weibullove porazdelitve. Podatki za primer so menjalni tečaji dolarja in japonskega jena ter britanskega funta in japonskega jena. Podatke logaritmiramo in neničelne spremembe tečaja modeliramo z asimetrično Weibullovo porazdelitvijo. Že razsevni grafikon sam pokaže, da je v podatkih prisotna asimetrija. Preverjanje prilagajanja enorazsežnih porazdelitev s QQ grafikoni pokaže, da je ujemanje dosti boljše kot pri privzetku o normalnosti. Ker so linearne kombinacije komponent asimetričnega Weibullovega vektorja tudi asimetrične Weibullove, lahko na enak način preverjamo tudi ujemanje poljubnih ortogonalnih projekcij na enotske vektorje v $\mathbb{R}^2$. Tudi QQ grafikoni za posamezne smeri pokažejo dobro ujemanje in so občutno boljši kot bivariatna normalna porazdelitev. Številne empirične študije so pokazale, da so porazdelitve finančnih podatkov tipično asimetrične s težkimi repi. Bivariatna Weibullova porazdelitev se je izkazala kot primernejši in zanesljivejši model za menjalne tečaje, kar nakazuje na uporabnost nove družine porazdelitev tudi v drugih okoliščinah.