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MASTER'S THESIS

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APPLICATION OF OPTIMAL CONTROL THEORY IN DYNAMIC OPTIMIZATION AND ANALYSIS OF PRODUCTION - INVENTORY CONTROL MODEL WITH QUADRATIC AND LINEAR COST FUNCTION

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Izjava

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Podpis:_________________________________

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1. INTRODUCTION

In the second half of the last century there was a rapid spread of scientific disciplines in rationalization of management decisions, a movement represented by terms "operations research" and "management sciences". It was a reflection of continuous interaction between a more careful definition of business problems and development of mathematical tools for their solution.

One of the consequences resulting from these movements was the rise of scientific area named Supply Chain Management (SCM), since it became necessary to develop a set of techniques for improving firm competitiveness by improving efficiency at the level of the channel rather than at the firm level. This is so because the real competition is not a matter of one company against another company but rather, one supply chain against another supply chain in achieving cost effective superior service [15].

The issue of production and inventory control is an essential part of SCM, and it is in this area that mathematical tools have been widely implemented.

Goals and purposes of the work

Since the dynamic optimization methods are among the most important and interesting optimization methods today, I choose the Optimal Control Theory, as a representative of these, for optimization of the production-inventory model, created as an upgrade of HMMS type models.

The results of optimization, the optimal solutions, considerably depend on the specification of objective function or functional. The HMMS type models are models that, in their goal functional, consider the costs of deviations of the actual inventory level and production rate from respective goal values. I modified this assumption so that, beside these types of costs, I also introduced linear costs for producing a unit of product and for keeping it in inventory stock and the discount rate. I did so because I consider that the costs of producing a unit of product or keeping it in the inventory stock are basically very different from the costs of production or inventory deviation from the desired level since they result from different causes, and hence their meaning is essentially different. It was by introducing such other types of costs that I tried to improve the model by getting it closer to reality. I also implemented a discount rate. In a control model, where the relevant time horizon is short, discounting is not important and it can be disregarded, but its role comes to effect when the planning period of a firm extends into the far future or even tends to infinity. The discounted returns of the very far future then become negligible. Introduction of discount rate means taking the opportunity costs into consideration, or the costs of loosing potential income that could have been realized with the money spent on inventory or production.

Since the purpose of developing, optimizing and analysis of mathematical models in any field of science is a better comprehension of the processes described by them, thus providing considerable help in making management decisions, I consider that understanding and proper interpretation of variables and basic equations, of which a model is built, is very important for providing a better insight of all its components. Therefore, in Chapter eight, I presented their economic interpretation.

In the first chapters, I introduced the problem of SCM and the role of production and inventory as its part. Then I presented a brief historical overview of some relevant models and several very important and widely used contemporary methods of managing production and inventory logistics. Since I used mathematical theory of optimal control to optimize the model, I wrote a short review of the models that were optimized using this theory. After that, I presented a basic explanation of three main methods for dynamic optimization and compared them to each other in order to explain why I choose very this optimal control theory and the Pontryagin's principle of maximality.

2. THE PROBLEM OF INVENTORY CONTROL MANAGEMENT IN ECONOMY, IN THE CONTEXT OF SUPPLY CHAIN MANAGEMENT.

Let us begin this chapter with explanation of what a supply chain (SC) is. It is a network of organizations that are involved in different processes and activities producing a value in the form of products and services in hands of ultimate consumers [15].

The most important activities among them are procurement of materials, their transformation into intermediate and finished products, distribution of products to consumers and recently, recycling of used goods as well.

So, the concern of supply chain management (SCM) is the control and management of material and information flow through a supply chain, from supplier to customer, in order to ensure that the right goods are delivered in the right place and quantity at the right time.

One definition says that supply chain management is a collection of functional activities that are repeated many times throughout the process, in which raw materials are converted into finished products [6].

The SCM consists of several areas such as forecasting, procurement, production, distribution, inventory, transportation, customer service and recycling. These can be viewed from different perspectives, for example, strategic, tactical and operational. Historically, these areas have been managed independently and buffered by large inventories.

Since the total investments tied in inventories are enormous and since they play a key role in the logistic behavior of production systems, the top management in many organizations has become aware that its efficient management (called supply chain inventory management - SCIM) is vital for the success of company.

SCIM can be defined as an integral approach to planning and control of inventory throughout the entire network of co-operating organizations, from source of supply to end-user [23]. It is focused on the ultimate customer's demand through improvement of customer service and decreasing of costs [39].

It is the problem of strategic importance for more or less all organizations in any sector of economy.

There are many reasons why organizations have to maintain inventories of goods. The most important one is that it is usually impossible to procure goods instantly when they are needed, because almost always there is a lead time between the ordering time and the delivery time. So, without an inventory on stock, the customer (or production) would have to wait. But since they do not want to or cannot be allowed to wait for long periods of time, the organizations would suffer loss. There are, of course, many other reasons for holding inventories. For example, due to certain ordering costs it is often necessary to order in batches instead of unit by unit, or the price of raw material for production may exhibit huge fluctuations, procurement of material in small batches is more expensive than in large ones and many other reasons.

The purpose of an inventory control system is, in general, to reduce holding and ordering costs while still maintaining satisfactory customer service.

It means that the objective of inventory control is therefore normally to balance conflicting goals. One goal is keeping stock levels down to make cash available for other purposes. The purchasing manager may wish to order large batches to get volume discounts. The production manager similarly wants long production runs to avoid time-consuming setups. He also prefers to have a high stock of finished goods to be able to provide customers with a high service level. It is not easy to find the optimal balance between such goals and that is why we need the inventory models.

There are many definitions of the area under the heading of inventory. For example: "An inventory is a stock of goods which is held or stored for the purpose of future sale or production " [3] or "the inventories are idle goods in storage waiting to be used [24] or [13] "the inventories are goods owned by economic subject. It means that:

1. inventories are goods
2. these goods are composed of a set of elements
3. these goods have an owner
4. these goods have to be considered from economic viewpoint

Some methods and models from that area address holding inventories, others address relation between controlling inventories and production in parallel, because these influence one another. For example, holding inventories has impacts on production policy by loosening the relation between production and sales. There is also an interaction between the present and future policies.

Basically, inventories in the inventory-production models constitute an alternative to production in the future. To have available one unit of product in the future, it may be either produced (or purchased) that time, or produced today and stored until the moment it is needed. The decision depends on the relative profitability of these two options. If an inventory is held, there are storage costs and capital tied in the inventory, which capital could have been invested somewhere else. On the other hand, it may happen that production of tomorrow is more expensive than production of today due to any of several reasons.

Anyway, it is obvious that a balance between production rate and inventory stock level is crucial for the proper functioning and the prospect of an organization.

3. HISTORICAL OVERVIEW OF SOME RELEVANT DEVELOPMENTS IN INVENTORY THEORY

3.1 Introduction

The inventory problems are as old as history itself, but the use of mathematical tools and analytical techniques in studying and managing these is dated back to the beginning of the 20th century. Reason for this was a rapid growth of manufacturing industries and various branches of engineering, especially industrial engineering. The necessity for this type of analysis first became prominent in industries that were facing a combination of production scheduling problems and inventory problems. For example, in organizations where items were produced in lots and stored in factory warehouses.

Two are fundamental questions that models for controlling inventories and physical goods must answer:

1. When to replenish inventory?
2. How much to order or produce for replenishment?

The procedures for determining these two quantities are called "lot sizing". The first and best known application of mathematical optimization methods for solving such a problem is the EOQ (Economic Order Quantity) model and so called "square root" or "Wilson lot-size" formula, developed by Harris in 1913. This model has been widely studied and it is a staple of almost every basic textbook addressing production and operations management. Because of its importance, I will present its result and the basic key insight of it. By doing so, in the same time, I will mention the principal terms used in it as well as in almost every study of inventory and production.

3.2 Insight of the EOQ model

The problem that Harris was concerning while developing this model was following: A factory produces various products and switching between products entails an expensive setup. If the products are produced in large lots, setup costs are reduced by less frequent changeovers. However, small lots reduce inventory by bringing the product closer to the time it is used. The EOQ model provided a systematic approach for handling balance between these two issues.

He used the following notation:
S = demand rate
p = unit production cost, not counting setup or inventory costs
A = fixed setup cost to produce a lot
h = holding cost (if the holding cost consists entirely of interest on money tied up in inventory, then h = ip, where i is the annual interest rate)
Q = lot size

The assumptions made in the model were as follows:
1. Production is instantaneous
2. Delivery is immediate
3. Demand is deterministic
4. Demand is constant over time
5. A production run incurs a fixed setup cost
6. Products can be analyzed individually

First, he derived the total cost per year (inventory, setup and production) as:

$$Y(Q) = \frac{hQ}{2} + \frac{AS}{Q} + pS$$

and from it, he deduced the economic order quantity (or lot size) as:

$$Q^* = \sqrt{\frac{2AS}{h}}$$

The obvious implication of the above result is that the optimal order quantity increases with the square root of the setup cost or the demand rate, while it decreases with the square root of the holding cost.

A more fundamental insight from Harris`s work is that there is a trade-off between lot size and inventory. Increase of the lot size increases the average amount of inventory on hand, but reduces the frequency of ordering.

One important characteristic of the EOQ model is that the total cost is rather insensitive to lot size. This means that if a slightly different lot size than Q^* is used, the increase of the holding plus setup costs will not be large.

Since the demand is assumed to be deterministic, using the determined order quantity, it is easy to determine the order interval as:

$$T = \frac{Q}{S}$$

It is important to notice that EOQ model was a static one because it assumed static demand.

3.3 Review of some important models from the past

Since the work of Wilson, the study of inventory systems has been a major concern in management science. A variety of different situations have been modeled, using a variety of different mathematical techniques. It would be fair to say that almost entire kit of modeling tools has been used sooner or later in solving different interpretations of inventory problems. It is important to notice that use of inventory models is not limited to controlling inventory in a warehouse, but it is widely implemented in a number of areas. For example, a forest control, pollution problems or a cash management, and so on. Types of inventory problems considered are also numerous. The variety can be highlighted by considering a set of diversities, which describes some of the approaches used. There have been continuous time and discrete time models, deterministic and stochastic, static and dynamic, periodic and non-periodic and so on. It is possible to find almost any combination of these characteristics in the past models.

It is therefore obviously impossible to mention all inventory models of interest. I have tried to choose the models that are important and that brought some paradigm into inventory modeling.

Harris`s original formula has been extended in a variety of ways over the years. One of the earliest extensions (Taft, 1918.) was to the case in which replenishment is not instantaneous; instead, there is a finite, but constant and deterministic, production rate. This model is sometimes called the economic production lot (EPL) model and results in a similar square root formula to the regular EOQ.

An important work, published 40 years after the EOQ, was a stochastic version of the simple lot size model, developed by Whitin. It was the first book in English addressing in detail the stochastic inventory models [26].

Another important work was the paper published by economists Arrow, Hariss and Marschak [1]. They were the first who provided a rigorous mathematical analysis of a simple type of inventory model. Their work was followed by often quoted and rather abstract papers by mathematicians Dvorezky, Kiefer and Wolfowitz [26].

An interesting and important book addressing mathematical properties of inventory systems was published by Arrow Karlin and Scarf [2] in 1958.

The main historical approach to relaxing the constant demand assumption is the article written by Harvey M. Wagner and Thompson M. Whitin [40] in 1958. They developed a forward algorithm for the solution of a dynamic version of the economic lot size model, allowing the possibility of demands for a single item, inventory charges, and setup costs to vary over N periods to get a minimum total costs inventory management scheme satisfying the given demand in any period. They used the methodology of discrete dynamic programming. Dynamic lot-sizing approach was important because of its substantial impact on the literature addressing production-inventory control as well as because of the influence it had on the development of Material Requirements Planning (MRP) later.

Other variations of the basic EOQ include backorders (i.e., orders that are not filled immediately, but have to wait until stock is available), major and minor setups, and quantity discounts (see Johnson and Montgomery 1974; McClain and Thomas 1985; Plossl 1985; Silver, Pyke, and Peterson 1998).

Another very important production-inventory control model was developed by Holt, Modigliani, Muth and Simon in their book [27]. It was named after the authors - HMMS model. Their model minimized the production and inventory costs in a continuous time by minimizing the quadratic deviations of inventory level and production rate from the respective goal values. They solved the model by calculus of variation techniques.

3.4 Brief review of optimal control models in production-inventory theory

In years following its occurrence, HMMS model inspired many optimal control theory formulations of production and inventory planning problem.

Among these, an important one was created by Hwang, Fan and Ericson [29], who introduced the maximum principle in their model. The advantage of optimal control theory formulation of HMMS model lies in the simple implementation of constraints on production rate.

Another advantage is simpler extension to multi-item production, done by Bergstrom and Smith [10] in 1970.

In 1972, Bensoussan [7] presented generalized optimal control theory formulation of continuous type, in which he tried to include several types of HMMS models. Two years later, G. Hurst and Jr. B. Naslund published a book [8] in which they also presented several types of models. First, a simple deterministic production control model with time-varying demand rate. Then they constrained the inventory and production levels as non-negative and summarized extensions to an observations about this common problem. Next, they considered the inventory problem, where orders are received as impulses rather than as smooth flows from production. They concluded with stochastic extension of the production planning model of HMMS model.

Similar type of model, also using methodology of optimal control theory, was presented in detail by Thompson and Sethy [37] in their book.

4. REVIEW OF SEVERAL IMPORTANT CONTEMPORARY METHODS OF MANAGING PRODUCTION LOGISTICS

4.1 Introduction

In the previous chapter, I presented an insight of EOQ model as a most widely studied and used model in the past. Now, I will present a contemporary method that is known and applied at least as much as EOQ or even more, and it has become the principal production control paradigm in the last three decades. It is also known as the first major implementation of computers in production control. It is named materials requirement planning (MRP) and it can be described as a management information system for determining production schedules in multilevel manufacturing system [25]. One of its most prominent features is well illustrated in the statement made by its author Joseph Orlicky, who said: "*Unlike many other approaches and techniques MRP works, which is its best recommendation*". MRP was developed in late 1960. It started slowly in the beginning, but experienced a tremendous boost in 1972, and together with its successors, manufacturing resources planning - MRP II and enterprise resources planning (ERP), has still been in use nowadays.

4.2 The key insight of MRP

In most of the production control systems, the need for producing or purchasing of an item of product (the demand) arises from the number of parts falling bellow some determined level. This approach suits better to the final product than to its components because the need for final product comes from outside the system; yet, since the components are used to produce final product, need for them is coming from inside the system. With the known or predicted demand for final product, called independent demand, the demand for components, called dependent demand, can be completely determined.

The main characteristic of MRP is that it considers the relationship between these two demands, which leads to a higher efficiency in scheduling production. It is fair to say that the basic mechanism of MRP schedules the production to meet the dependent demand so that it explicitly acknowledges its linkage to the production that has to meet independent demand.

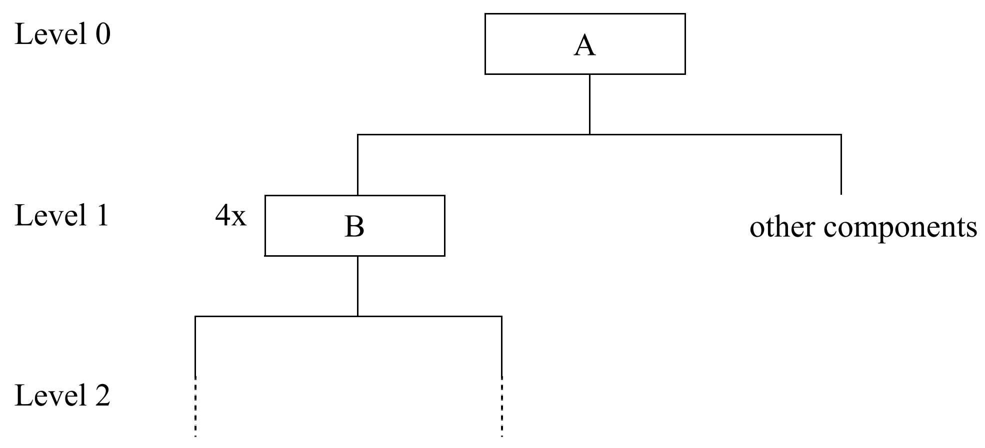
4.3 Overview of MRP

MRP deals with two basic dimensions of production control: quantities and timing.

For both types of items, final product and components, it determines respective required production quantity and production timing, for example, start time of production in order to meet order due dates. The length of time for which production is planned is called bucket. The number of periods (buckets) considered is called planning horizon. In the past, buckets usually equaled to a week or longer, but as data processing became considerably less expensive, buckets are becoming shorter.

The exact relationship between the final products (end items) and components (lower-level items) is defined with the bill of material (BOM). Figure 1 provides an example of BOM for the item (final product) A, where, for example, four pieces of items (components) B and other low-level items are needed.

Figure 1: *Bill of material (BOM)*



Along with BOM, another important information that MRP uses is referring to independent demand, which is contained in the master production schedule (MPS). It contains three types of information: gross requirements for final product, gross current inventory status called on-hand inventory and the status of both, purchasing and manufacturing orders, known as scheduled receipts.

The basic procedure of MRP can be described by following operations (steps):

First, it determines net requirements for the final product (or zero level of BOM) by subtracting on hand inventory and any scheduled receipts from the gross requirements of MPS.

Second, it divides the demand into appropriate lot size using one of the lot-sizing methods (lot-for-lot, POQ, EOQ, Wagner-Whitin method, Silver-Meal heuristic or other)

Third, it determines the start times for production using due dates and lead times.

Fourth, using the start times, lot sizes and BOM, it generates gross requirements of all required components for the next level.

All steps are repeated for each of these components, but using the new gross requirements generated in the previous step as the input instead of the one obtained from MPS.

4.4 Brief outline of two important successors of MRP

As it was shown, the MRP contains a method for planning and procuring the materials to support production.

During years of using MRP, the need for other functions arose that would, together with MRP, create an actually integrated manufacturing management system .

It was done by creating a large production control system named manufacturing resources planning (MRP II). The main additional functions contained in MRP II were: demand management, forecasting, capacity planning, master production scheduling, rough-cut capacity planning, capacity requirements planning, dispatching and input/output control. Along with integrating all of these additional functions in one whole**,** it also implemented the procedures developed during the years of using MRP to solve some problems that appeared in MRP. It resulted in a general control structure that breaks the production control problem into segments and provides a hierarchical approach [28]. The exact hierarchy of the MRP II defers from one software package to another.

In the years following the development of MRP II, there have been a few not so successful tries to improve MRP II, such as MRP III and business requirement planning (BRP) until another very successful successor, named enterprise resource planning (ERP) has occurred. It integrated the hierarchical approach of MRP II into a remarkable management tool that is capable to manage enormous quantity of data [28]. The main advantage of ERP was, in fact, that it linked together all information facilitating top management to see all operations of the system more globally and in real time. In brief, it could be said that it offered the instant control of the entire enterprise, and it was named after this feature. One source of the idea for developing such a system was the recognition and appearance of a new area, named supply chain management, which extended traditional inventory control methods to an integrated approach of the planning and control of the entire network of functions to include forecasting, procurement, production, distribution, inventory, transporting, customer services and even recycling. It is the definition of this field that made logistic issues of strategic importance and hence the system that supports and enables decision making on that level, such as ERP, became unavoidable.

5. ANALYSIS AND COMPARISON OF DIFFERENT DYNAMIC OPTIMIZATION METHODS WITH PARTICULAR EMPHASIS ON OPTIMAL CONTROL THEORY

5.1 Problem of dynamic optimization

The most important goal for a model designer is to find constructive methods and techniques for determining optimal strategies, or, in other words, to optimize it, in order to be able to analyze its behavior and to make the right management decision for controlling it. That is why the optimization is one of the most important issues when analyzing systems in different fields of science and everyday life.

The classic calculus methods of finding free and constrained extrema and techniques of mathematical programming are very useful but they are applicable only to the static optimization problems [17]. The solution sought in such problems usually consists of a single optimal magnitude for every choice variable. If a system has to be controlled and managed optimally day after day or hour after hour or even continuously, tools for dynamic optimization must be used because this poses the question of what is the optimal magnitude of a choice variable in each period of time within a planning period (discrete time case) or at each point of time in a given time interval [0,T] (continuous time case). The solution of a dynamic optimization problem would thus take the form of an optimal time path for every choice variable.

Today, the dynamic system optimization is of the major interest in Management Science and therefore many authors address and explain it (such as [17, 22, 31, 32, 33, 34, 35, 37]). They gave an outline of different mathematical methods for solving it.

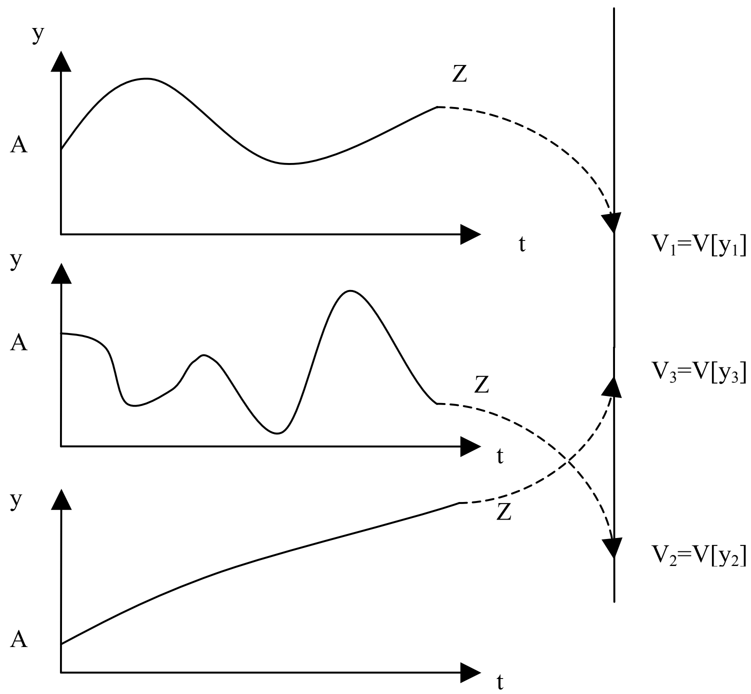
Regardless of whether variables are discrete or continuous, a simple type of dynamic optimization problem would contain the following basic components (A. C. Ciang, 1992, p. 6.):

- given initial point and a given terminal point
- set of admissible paths from the initial point to the terminal point
- set of path values serving as performance indices (cost, profit, etc.) associated with the various paths

- specified objective, either to maximize or to minimize the path value or performance index by choosing the optimal path

The relationship between paths and path values is a special sort of mapping, which is not an usual function but the mapping from paths (curves) to real numbers and it is called a functional. It is illustrated on Figure 2. This concept takes a prominent place in the dynamic optimization. Symbol V[y] or V{y} will be used for path values because it emphasizes the fact that its value depends on the change in position of the entire y path (the variation in the y path).

Figure 2: *Illustration of the concept of functional*



It is important to say that the initial and terminal points must not be fixed points. Since they consist of time and state, different situations may be considered. First, since the optimizing plan must start at some specific initial position, the initial point is often fixed, and here it will be considered so. Depending on the types of terminal point, problems can be sorted in the following different types [17]:

- fixed-time-horizon problem, meaning that the terminal time of the problem is fixed but the terminal state is free. This problem is often called the vertical terminal line problem. If the terminal state is not completely free but bounded from the top or bottom, it is called truncated vertical terminal line (Figure 3.a)

- fixed-endpoint problem (even there the whole endpoint is not fixed) where terminal time is free but the terminal state is fixed. It is also called a horizontal terminal line problem (Figure 3.b). It can also be truncated
- variable-terminal-point problem where neither the terminal time T nor the terminal state Z is individually preset, but these two are tied together via a constraint equation of the form Z=φ(T). It is also called a terminal-curve problem (Figure 3.c).

In all three types of the problems there is one more degree of freedom than in the fixed-terminal-point problems. For that reason, another condition, named transversality condition, that describes how the optimal path crosses (transverses) the terminal line or curve, is needed to determine the optimal paths.

Figure 3.a: *Vertical terminal line*

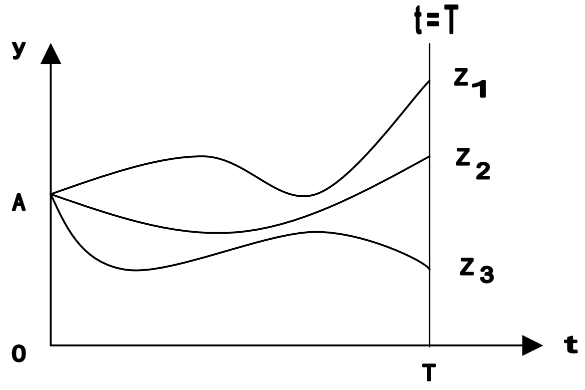


Figure 3.b: *Horizontal terminal line*

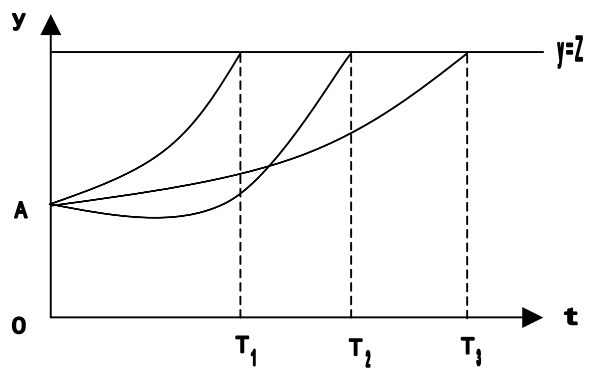
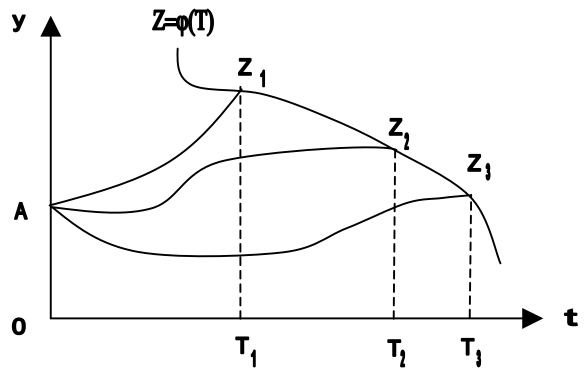


Figure 3.c: *Terminal curve*



5.2 The explanation of the meaning of objective functional

The optimal path is the admissible path that maximizes or minimizes the path value V[y]. Since any y path must travel through an interval of time, its total value in the discrete type problem is the sum, while in the continuous type problem it is the definite integral, of the form $\int_0^T (arcvalue)dt$. For arc identification, three information are needed: the starting stage (time) t, the starting state y(t), and the direction in where the arc proceeds $y'(t)=dy/dt$. So, the integral can be written as:

$$V[y] = \int_0^T F[t, y(t), y'(t)]dt \qquad (5.1)$$

A problem with an objective functional in the form of (5.1) is called the standard problem (A. C. Ciang, 1992, p. 13)

5.3 Three major approaches for solving dynamic optimization problems

Three most important algorithms for solving the dynamic optimization problems are:

1. Calculus of variations [CV]
2. Optimal control theory [OCT]
3. Dynamic programming [DP]

All three theories are very closely related, and, under certain differentiability assumptions, one can be deduced from the other.

5.3.1 The Calculus of Variation

Back in the late 17th century, the calculus of variations was the classical approach to the problem. The fundamental problem (or the simplest problem) of calculus of variation is represented by the following general formulation (A. C. Ciang, 1992, p. 27.):

Maximize or minimize $V[y] = \int_0^T F[t, y(t), y'(t)]dt$

subject $y(0) = A$ (A given) (5.2)

and $y(T) = Z$ (T,Z given)

The methods used in the calculus of variations closely parallels to and extends techniques of point optimization in differential calculus into the function space. The basic difference is that here the problem is to determine, under certain conditions, an optimum function (path) instead of optimum point [35]. A smooth optimum function (path) that yields an extrem value (maximum or minimum) of V[y] is called an extremal.

The basic first-order necessary condition in the calculus of variation is the Euler equation:

$$F_y - \frac{d}{dt}F_{y'} = 0 \quad \text{for all } t\in[0,T] \qquad (5.3)$$

or it can be represented in the form:

$$\int F_y dt = F_{y'} \qquad (5.4)$$

(which is the result of integrating (5.3) with respect to t) or in more explicit form:

$$F_{y'y'}y''(t) + F_{yy'}y'(t) + F_{ty'} - F_y = 0 \quad \text{for all } t\in[0,T] \qquad (5.5)$$

It can be easily generalized in two ways:

For the case of n>1 state variables. Then there will be a pair of initial conditions and a terminal condition for each of the n state variables and the single Euler equation will be replaced by a set of n simultaneous Euler equations.

For the case of higher order derivatives. In this case there will be 2n boundary conditions and the F function can be transformed into an equivalent form containing n state variables and their first-order derivatives only.

It was mentioned before that transversality condition must exist for the problems not having a fixed end point. The role of transversality condition is to substitute a missing terminal condition:

$$[F - y' F_{y'}]_{t=T} \Delta T + [F_{y'}]_{t=T} \Delta y_T = 0 \tag{5.6}$$

The conditions shown till now were only the first-order necessary conditions and they only served to identify the extremals of the problem without consideration as to whether they maximize or minimize the functional V[y]. The sufficient condition for fixed endpoint problem from [17] is given with:

Sufficiency theorem

For the fixed endpoint problem (1.2) if the integrand function F(t,y,y') is concave (jointly) in the variables (y,y'), then the Euler equation is sufficient for an absolute maximum of V[y]. Similarly, if F(t,y,y') is convex (jointly) in the (y,y'), then the Euler equation is sufficient for an absolute minimum of V[y].

It is worthwhile to introduce a second-order necessary condition, known as the Legendre condition, which is based on local concavity/convexity. It is not as powerful as the sufficient condition but it is very useful.

$$\begin{aligned} &\text{Maximization of V[y]} \quad \Rightarrow \quad F_{y'y'} \le 0 \quad \text{for all } t \in [0,T] \\ &\text{Minimization of V[y]} \quad \Rightarrow \quad F_{y'y'} \ge 0 \quad \text{for all } t \in [0,T] \end{aligned} \tag{5.7}$$

The $F_{y'y'}$ derivative is evaluated along the extremal.

The calculus of variations can be applied even to the problems with infinite planning horizon, by introducing some methodological changes as well as to the constrained problems by using Lagrangian integrand function (which is not exactly the same as the one used in the static optimization).

The main failure of calculus of variation is that it cannot handle corner solutions (boundary) and the objective function that is linear in y(t).

5.3.2 Optimal Control Theory

This is a new approach to the dynamic optimization, which is actually an outcome of calculus of variations (Sargent, 2000. p. 361.). (Sethy, 1999. p. 1.) defines it as a branch of mathematics developed to find the optimal ways for controlling a dynamic system.

In optimal control theory, the dynamic optimization problem has three instead of two types of variables like in CV. A new, control variable u(t) is introduced, and it serves as an instrument of optimization. It is this variable that the theory has been named after. Since the attention is focused on the control variable, once the u(t) is found, and using an initial condition on y(t), it must determine a state variable path as by-product. This is why an optimal control problem must contain an equation that relates y(t) to u(t), which is called the equation of motion (or transition equation or state equation) $\frac{dy}{dt} = f[t, y(t), u(t)]$. The optimal control problem corresponding to the calculus of variations problem (5.2) is as follows:

$$\text{Maximize or minimize} \quad V[u] = \int_0^T F[t, y(t), u(t)]dt$$

$$\text{subject to} \quad \dot{y}(t) = f(t, y(t), u(t))$$

$$y(0)=A \qquad \text{(A given)} \tag{5.8}$$

$$\text{and} \quad y(T)=Z \qquad \text{(T,Z given)}$$

Since the state variable y(t) is determined as by-product of decision variable u(t), the situation with given terminal value is no more the simplest one as it was in CV.

The simplest problem of optimal control has a free terminal state (vertical terminal line):

Maximize $V[u] = \int_0^T F[t, y(t), u(t)]dt$

subject to $\dot{y}(t) = f(t, y(t), u(t))$ (5.9)

$y(0)=A$ $y(T)$ free (A,T given)

and $u(t) \in U$ for all $t \in [0,T]$

5.3.2.1 An outline of the maximum principle

The most important and best known development in optimal control theory (a first-order necessary condition) is called maximum principle. It was independently developed by two authors in the same time: Russian mathematician L. S. Pontryagin and American mathematician Magnus R. Hestenes.

The statement of the maximum principle involves the concepts of the Hamiltonian function and costate variable (or auxiliary variable), which is denoted by λ. It is akin to Lagrange multiplier and as such it is in the nature of a valuation variable, measuring the shadow price of an associated state variable. Like y(t) and u(t), this variable can get different values at different points of time so the λ is a short version of λ(t). The carrier by which the costate variable enters into the optimal control problem is the Hamiltonian function, or simply the Hamiltonian denoted by H [17]. It is defined as

$$H(t,y(t),u(t),\lambda(t)) \equiv F(t,y(t),u(t)) + \lambda(t)f(t,y(t),u(t)) \qquad (5.10)$$

Maximum principle involves two first-order differential equations: in the state variable y(t) and the costate variable λ(t). Besides, there is a requirement that the Hamiltonian is maximized with respect to the control variable u(t) at every point of time. For the problem in (5.9) and with the Hamiltonian in (5.10) the maximum principle conditions are:

$$\underset{u}{Max}\, H(t, y(t), u(t), \lambda(t)) \text{ for all } t \in [0,T]$$

$$\dot{y} = \frac{\partial H}{\partial \lambda} \qquad \text{(equation of motion for y)}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial y} \qquad \text{(equation of motion for } \lambda\text{)}$$

$$\lambda(T) = 0 \qquad \text{(transversality condition)}$$

When the Hamiltonian is differentiable with respect to u, it gives an interior solution, so the condition $\frac{\partial H}{\partial u} = 0$ (supported by an appropriate second-order condition) can be used for maximization. Otherwise, since the set of admissible paths may be a closed set with possible boundary solutions, the broader statement $\underset{u}{Max}\, H(t, y, u, \lambda)$ for all t $\in$[0,T] or $H(t, y(t), u^*(t), \lambda(t)) \geq H(t, y(t), u(t), \lambda(t))$ for all t$\in$[0,T] is needed. It is important to point out that, in order to apply the maximum principle, the Hamiltonian is not even required to be differentiable with respect to u(t).

These two equations of motion together are collectively referred as the Hamiltonian system, or the canonical system. In some models, it turns out to be more convenient to deal with a dynamic system in the variables (y(t),u(t)) in place of the canonical system in the variables (y(t), λ(t)).

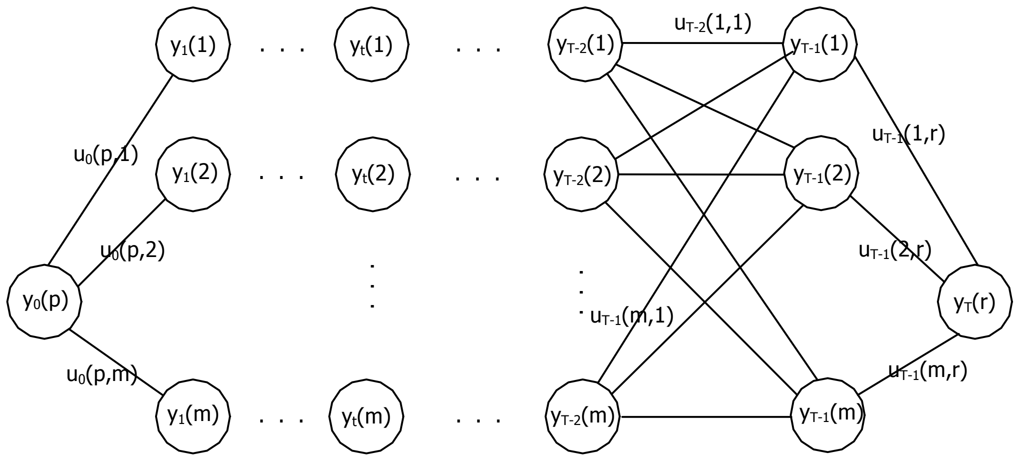
5.3.3 Dynamic Programming

Dynamic Programming was pioneered by the American mathematician Richard Bellman, who, in his book under the same title, published in 1957, presented another approach to control problem stated in (5.8).

Two most important characteristics of this approach are: first, that it 'embeds' the control problem in stages, which are all treated as an individual control problem; second, for each of these new control problems, the central attention is on the value of the functional (V^*)

To illustrate and explain basic reasoning of DP, I will draw a graph for the discrete case (Figure 4). Nodes (circles) in the graph present states of the system, and arcs between them present the value of decision variable (control), that alters system state from one to another. The values of these decision variables will be considered as costs, which have to be minimized. Let the system has, for example, m possible states. They will be marked as $y_t(j)$ $(j = 1,...,m)$ where t denotes the stage of the problem (t=0,...,T). The value (cost) of decision to bring the system from state $y_{t-1}(i)$ of stage t-1 to state $y_t(j)$ of stage t, is denoted by $u_{t-1}(i,j)$.

Figure 4: *Discrete dynamic programming scheme*



Referring to Figure 4, the "embedding" of a problem is performed as follows:

Instead of the primary given problem of finding the minimum-cost path from the chosen initial state $y_0(p)$ of stage 0 to the chosen terminal state $y_T(r)$, a broader problem of finding the minimum-cost path from each point in the graph to the desired terminal point $y_T(r)$ has to be considered. Every component of the initial problem now has its own initial point.

Since every component of the problem has a unique optimal (minimum-cost) path value, it is possible to write an optimal value functions:

$$V^* = V^*(y_t(j)) \qquad \text{(t=0,1, ... ,T-1) -stage} \quad \text{(j=1, ... ,m)}$$

showing that an optimal path value can be determined for every possible initial point. Using these, an optimal policy function can also be constructed to show the optimal path from any specific initial point $y_t(j)$, in order to achieve $V^*(y_t(j))$ by the proper selection of a sequence of arcs from point (state) $y_t(j)$ to the terminal point $y_T(r)$.

The purpose of embedding process is to develop an iterative procedure for solving the initially given problem.

The first problem is to determine the optimal values for stage T-1, associated with the initial points $y_{T-1}(j)$ (j=1, ... ,m) given by:

$$V^*(y_{T-1}(j))=\min\{u_{T-1}(j,r)\} \qquad (j=1, \dots ,m) \qquad (5.11)$$

When these are found, the next step is to find the optimal (minimum-cost) values $V^*(y_{T-2}(j))$ (j=1,...,m) for the previous stage. Utilizing the previously obtained optimal-value information in (5.11), if I choose k-th point (state) in stage T-2 its optimal value function $V^*(y_{T-2}(k))$, as well as the optimal path (decision) $u_{T-2}(k,j)$ (j=1, ... ,m), can be determined as follows:

$$V^*(y_{T-2}(k)) = \min\{u_{T-2}(k,j) + V^*(y_{T-1}(j))\} \qquad (5.12)$$

When the outgoing arcs (paths) from stage T-2 to stage T-1 (which are parts of the optimal path from stage T-2 to T) are found for each point of stage T-2, they should be marked by asterisk. Proceeding backwards with the recursion equation (5.13) and marking the arcs, once the stage 0 has been reached, both optimal path and optimal value function of the first point (stage 0) are found, which is actually the solution of the originally given control problem.

$$V^*(y_{t-1}(k)) = \min\{u_{t-1}(k,j) + V^*(y_t(j))\} \quad (j=1,\dots,m) \qquad (5.13)$$

It can be concluded that the dynamic programming for discrete case is an iterative procedure, whose essence is contained in Bellman's principle of optimality. It could be briefly interpreted as follows: if the first arc, from an optimal sequence of arcs, is cut off, the sequence of remaining arcs must still be optimal path for respective initial point.

5.4 Comparison of different dynamic optimization methods

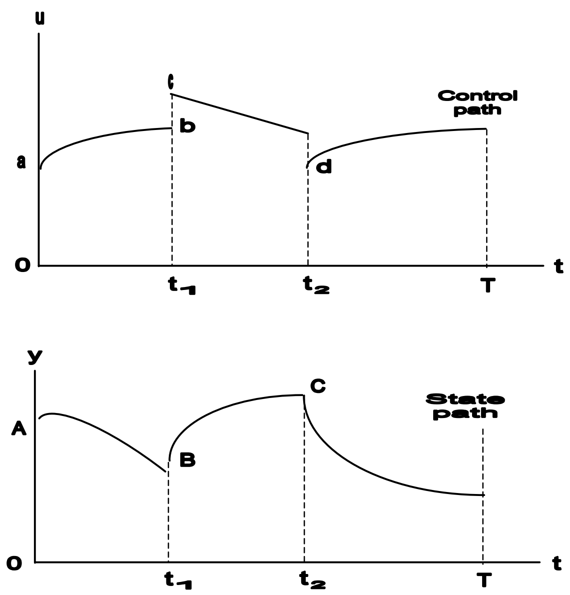
The method of dynamic programming, explained in the chapter 5.3.3 was of discrete type, and this type is often used in practical applications. The full version of DP also includes a continuous-time case, but its important feature is that the solution involves mathematical topics of partial differential equations and it is often not possible to find the analytical solution. The advantage of the other two methods (CV and OCT) is that they require only ordinary differential equation for their solutions, and this is why they are used more often for such type of problems.

Further characteristics making distinction among these three methods for dynamic optimization are as follows:

- in solving particular steps of a dynamic programming methods, the primary focus is on the optimal value of the functional
- in the calculus of variation, the focus is on the properties of the optimal state path, and
- in the optimal control theory, the focus is on the optimal control path.

The next important difference is between CV and OCT. For its applicability, the calculus of variations requires the differentiability of the function in the problem, and only the interior solution can be handled. The optimal control theory, on the contrary, can handle non-classical features, such as corner solutions. A control path does not have to be continuous to be admissible, it only needs to be piecewise continuous. This means that it is allowed to contain jump discontinuities. A state path only needs to be piecewise differentiable.

Figure 5: *Relation between control and state path in OCT*



Each sharp point on the state path occurs in the same moment in which control path has a jump (Figure 5).

Another important feature of optimal control theory is capability to handle a constraint on the control variable u(t) directly. It allows studying problems where admissible values of the control variable u(t) are confined to some closed bounded convex set U (u (t) ∈ U for 0 ≤ t ≤ T). Since U can be a closed set, corner solution (boundary solution) are possible and they can be handled. In this light, the control problem (5.8) constitutes a special (unconstrained) case where the control set U is the entire real line.

Another feature that should be noticed is that unlike the Euler equation of CV, which is a single second-order differential equation in the state variable y, maximum principle involves two first-order differential equations: in the state variable y and the costate variable λ.

Finally, it could be pointed out that the simplest problem in optimal control theory, unlike in the calculus of variations, has a free terminal state (vertical terminal line) rather than a fixed terminal point.

5.5 The rationale of developing the maximum principle by variational view

In Chapter seven, I will present the economic interpretation of some important variables and functions used in the maximum principle applied on the model presented in this paper. For that purpose, the rationale of developing the maximum principle needs to be explained and made plausible. It will be presented here by variational view using the same notation used further in the model. For the sake of plausibility, I will omit argument t in writing.

P - Production rate at time t (control variable)
I - Inventory level at time t (state variable)
F - Underintegral function (in presented model it will be the cost function with negative sign)
J - Functional that has to be maximized

To make it simpler, the control variable P will be assumed unconstrained, and since Hamiltonian in the model presented later is differentiable with respect to P, the condition

$$\frac{\partial H}{\partial P} = 0 \tag{5.5.1}$$

can be used for maximization, instead of the broader condition: $\underset{P}{Max} H$.

The initial point is fixed, and terminal point can vary. Therefore, the transfersality condition is needed and it will be derived by the way. The problem is given by

$$Max J = \int_0^T F(t, I, P)dt$$
$$\dot{I} = f(t, I, P) \tag{5.5.2}$$
$$I(0) = I_0 \quad \text{-given}$$

Using the notation of Lagrange multipliers the following expression will be introduced

$$\lambda(t)[f(t, I, P) - \dot{I}] \tag{5.5.3}$$

and its integral $\int_0^T \lambda(t)[f(t, I, P) - \dot{I}]dt$ will be added to the objective functional. It can be done because it does not change the solution for the following reason: if the variable I always obeys the equation of motion, the expression (5.5.3) will have a zero value for every $t \in [0, T]$ and the value of integral will be zero. A new objective functional is then given by:

$$J' = J + \int_0^T \lambda(t)[f(t, I, P) - \dot{I}]dt$$
$$= \int_0^T \{F(t, I, P) + \lambda(t)[f(t, I, P) - \dot{I}]\}dt \tag{5.5.4}$$

Substituting the Hamiltonian:

$$H(t, I, P, \lambda) = F(t, I, P) + \lambda(t) f(t, I, P) \tag{5.5.5}$$

into (5.5.4) it will simplify it like this:

$$J' = \int_0^T [H(t, I, P, \lambda) - \lambda(t)\dot{I}]dt$$
$$= \int_0^T H(t, I, P, \lambda)dt - \int_0^T \lambda(t)\dot{I}dt \tag{5.5.6}$$

When second integral is integrated by parts it gives:

$$J' = \int_0^T [H(t, I, P, \lambda) + I\dot{\lambda}]dt - \lambda(T)I(T) + \lambda(0)I_0 \qquad (5.5.7)$$

As is was shown before the $\lambda(t)$ will have no effect on the value J' as long as the equation of motion from (5.5.2) is strictly obeyed or as long as

$$\dot{I} = \frac{\partial H}{\partial \lambda} \qquad \forall t \in [0, T] \qquad (5.5.8)$$

That is why (5.5.8) is imposed as a necessary condition for the maximization of J'. (This is mere equation of motion written in another way)

Now it could be imagined that the optimal path $P^*(t)$ is known. It can be perturbed with a perturbing curve p(t) creating a "neighboring" control paths

$$P(t) = P^*(t) + \varepsilon p(t) \qquad (\varepsilon \in R) \qquad (5.5.9)$$

This will cause the perturbation of state path I(t) in accordance to the equation of motion from (5.5.2) and will generate state paths

$$I(t) = I^*(t) + \varepsilon q(t) \qquad (5.5.10)$$

If T(t) and I(t) are considered variable, there are perturbations for them, too:

$$T(\varepsilon) = T^* + \varepsilon \Delta T \qquad (5.5.11)$$

and

$$I(T) = I^*(T) + \varepsilon \Delta I(T) \qquad (5.5.12)$$

From here, for the sake of plausibility, I will omit argument ε in writing T(ε). Using (5.5.9), (5.5.10), (5.5.11) and (5.5.12) the expression (5.5.7) can be transformed into

$$J' = \int_0^T \{H[t, I^* + \varepsilon q(t), P^*(t) + \varepsilon p(t), \lambda] + \dot{\lambda}[I^* + \varepsilon q(t)]\}dt - \lambda(T)I(T) + \lambda(0)I_0 \quad (5.5.13)$$

and the first-order condition $\frac{dJ'}{d\varepsilon} = 0$ can be applied. The differentiation of integral gives

$$\int_0^T \{\frac{\partial H}{\partial I} q(t) + \frac{\partial H}{\partial P} p(t) + \dot{\lambda} q(t)\} dt + [H + \dot{\lambda} I](T) \frac{dT}{d\varepsilon} \qquad (5.5.14)$$

and the derivative of the second term is

$$-\lambda(T)\frac{dI(T)}{d\varepsilon} - I(t)\frac{d\lambda(T)}{dT}\frac{dT}{d\varepsilon} = -\lambda(T)\Delta I(T) - I(T)\dot{\lambda}(T)\Delta T \qquad (5.5.15)$$

The derivative of the third term is zero. So the $\frac{dJ'}{d\varepsilon}$ is the sum of (5.5.14) and (5.5.15). The last part of (5.5.14) can be rewritten like this:

$$[H + \dot{\lambda} I](T)\frac{dT}{d\varepsilon} = H(T)\Delta T + \dot{\lambda}(T) I(T)\Delta T \qquad (5.5.16)$$

The changed (5.5.14) expression is added to (5.5.15) and that sum is equated to zero (the first-order condition). It finally gives an important expression:

$$\frac{dJ'}{d\varepsilon} = \int_0^T [(\frac{\partial H}{\partial I} + \dot{\lambda}) q(t) + \frac{\partial H}{\partial P} p(t)] dt + H(T)\Delta T - \lambda(T)\Delta I(T) = 0 \qquad (5.5.17)$$

The different transfersality conditions can be derived from this expression.

It is important to notice that in all three aditive components of the middle part of equation (5.5.17), different arbitrary components exist. In first component, there are perturbing curves p(t) and q(t), in second ΔT and in third ΔI(T). Consequently, factors multiplying each of these arbitrary elements must individually be set to zero. From the first (integral) component, two equations are deduced:

$$\dot{\lambda} = -\frac{\partial H}{\partial I} \qquad (5.5.18)$$

and

$$\frac{\partial H}{\partial P} = 0 \qquad (5.5.19)$$

The equation (5.5.18) is actually the equation of motion for the costate λ.

The second is the first-order condition for maximizing Hamiltonian but the weaker version because it includes the assumption of differentiability of Hamiltonian with respect to P.

Transfersality conditions for two different terminal points will be deduced from the second and the third term of the middle part of equation (5.5.17):

1. For the vertical terminal line

Since T is fixed and I(T) is free, it is obvious that ΔT is automatically equal to zero but ΔI(T) is not. So, for that case, the tranfersality condition must be imposed like this:

$$\lambda(T) = 0 \tag{5.5.20}$$

2. For the truncated vertical terminal line

In this case T is fixed and I(T) is not completely free, but it is subject to inequation $I(T) \geq I_{\min}$. So the optimal solution can have only two possible types of solutions: $I^*(T) > I_{\min}$ or $I^*(T) = I_{\min}$. In the former (since the terminal conditions are automatically satisfied) the previously given transfersality condition for vertical terminal line applies. So,

$$\lambda(T) = 0 \text{ for } I^*(T) > I_{\min} \tag{5.5.21}$$

Otherwise, when $I^*(T) = I_{\min}$ the terminal condition is binding and the admissible neighboring paths for I are only those whose terminal states are $I(T) \geq I_{\min}$. So, introducing $I^*(T) = I_{\min}$ in (5.5.10) gives:

$$I(T) = I_{\min} + \varepsilon q(T) \tag{5.5.22}$$

Assuming $q(T) > 0$, from the condition $I(T) \geq I_{\min}$, it can be concluded that $\varepsilon \geq 0$.

Following Kuhn Tucker conditions, it will change the first-order condition $\frac{dJ'}{d\varepsilon} = 0$ into $\frac{dJ'}{d\varepsilon} \leq 0$ for a maximization problem. Now (5.5.17) implies the inequality transfersality condition

$$-\lambda(T)\Delta I(T) \leq 0 \tag{5.5.23}$$

Also, from (5.5.22) it can be seen that with $\varepsilon \geq 0$ the condition $I(T) \geq I_{\min}$ in this situation implies $\Delta I(T) \geq 0$. So, (5.5.23) becomes:

$$\lambda(T) \geq 0 \qquad \text{for} \qquad I^*(T) = I_{\min} \tag{5.5.24}$$

Finally, when (5.5.21) and (5.5.24) are put together, they obtain the transfersality conditions for truncated vertical terminal line as a complementary-slackness condition from the Kuhn-Tucker conditions:

$$\lambda(T) \geq 0 \qquad I(T) \geq I_{\min} \quad (I(T) - I_{\min})\lambda(T) = 0 \tag{5.5.25}$$

I will conclude my presentation here because this will do for the purpose of economic interpretation of the model that is made in the Chapter eight.

6. DEVELOPMENT AND UPGRADING OF THE PRODUCTION INVENTORY CONTROL MODEL WITH QUADRATIC AND LINEAR COST FUNCTION

The model that will be developed here is a modification of the HMMS type model for a firm producing some homogenous goods and having a warehouse for inventory.

As mentioned before, the HMMS type models consider production and inventory holding costs over time in a way that costs are introduced only for the quadratic deviation of inventory level and production rate from the respective goal values.

I believe that the main drawback of such a pure quadratic criterial functional implemented in HMMS type models is that all costs are approximated only with the deviation costs. Because of that, in the situation where a manager keeps, for example, inventory at desired level, it would turn out that there are no costs for them at all, which is obviously an enormous deviation from reality. Of course, the same reasoning goes for production, too. This is why I introduced linear cost as well. It represents the sum of costs depending linearly (i.e., they are proportional to) on the amount of produced or stored goods. For example, the cost of material used for production of one unit, or cost of regular work hours needed for it. Or, in the case of inventory costs, it is, for example, the cost of maintaining the place in warehouse needed for a unit of item, or, for example, an opportunity cost for the amount of money equal to item worth, and so on. In the same time, under the costs of deviation of desired levels (quadratic costs), I assume only the extra costs, resulted exclusively from those deviations. For production higher than desired, it would be, for example, the difference between the price of normal work hour and overtime work hour. For production lower than desired, it could be the costs that a firm has to pay for even idle work hours. For the inventory that does not match the desired level, extra costs can be caused, for example, by maintaining an empty or half empty warehouse space, or because not being able to fulfill an order on time if inventory is lower. If it is higher, then a firm may have to hire some extra space to stock it, or if the goods are perishable, more goods could be absoleted, or if too many final products are on stock, it could also have an impact on reducing the price of goods, and so on. I will not go into details specifying all of the reasons that may cause these two types of costs.

The important issue I want to point out to here is that all costs occurring in a firm producing and storing goods (regardless of material type, i.e., raw materials, semi products, final products, or spare parts, or all of these) can be classified and approximated to these two types of costs, much easer and better than only in to the costs of deviations from the goal levels, as it is assumed in the HMMS type of models.

In order to make the model even closer to reality, I also introduced continuous discounting, because it allows long-time or even infinite planning horizon analysis.

6.1 Development and optimization of the model without constraints, with constant desired levels of inventory and production and finite planning horizon

The following quantities are needed for defining the model:

P(t) - Production rate at time t (control variable)

I(t) - Inventory level at time t (state variable)

ρ - constant, nonnegative continuous discount rate

$\hat{P}$ - constant, nonnegative, desired level of production

$\hat{I}$ - constant, nonnegative, desired level of inventory

a - constant, positive, extra inventory holding costs coefficient, which the firm has because of deviation of actual inventory from the desired level

b - constant, positive, extra production costs coefficient, which the firm has because of production deviation from the desired production

p - constant, positive, linear production cost coefficient for unit of product h - constant, positive, linear inventory holding cost coefficient for keeping unit of inventory

S(t) - positive, continuously differentiable, exogenous demand rate at time t

T - length of planning period

I_0 - constant, positive, initial inventory level

Change of inventory level follows the usual stock-flow differential equation:

$$f(t,I,P) = \dot{I}(t) = P(t) - S(t) \tag{6.1}$$

and initial condition $I(0) = I_0$.

6.1.1 Optimization of the model

The goal is to minimize costs that can be expressed by the objective function of the model

$$\max J = -\int_0^T e^{-\rho t}[a(I - \hat{I})^2 + hI + b(P - \hat{P})^2 + pP]dt \tag{6.2}$$

The expression

$$C(t,I,P) = a(I - \hat{I})^2 + hI + b(P - \hat{P})^2 + pP \tag{6.2'}$$

represents a cost function and the "underintegral" function F that enters the problem is discounted negative of that function, because the functional has to be maximized

$$F(t,I,P) = -e^{-\rho t}C(t,I,P)$$

For now, $\hat{P}$ will be assumed sufficiently large and I_0 sufficiently small, so that P will not become zero, and hence there are no constraints on P or I.

Since the objective function is discounted, it is convenient to use a current value formulation of maximum principle [37].

Instead of using standard (present-value) Hamiltonian with standard Lagrange multiplier (adjoint variable) λ_S that would look like this:

$$H_S = -e^{-\rho t}[-a(I-\hat{I})^2 - hI - b(P-\hat{P})^2 - pP] + \lambda_S(P-S)$$

a current value Hamiltonian, defined by $H \equiv H_S * e^{\rho t}$ with current value adjoint variable defined by $\lambda \equiv \lambda_S * e^{\rho t}$, will be used:

$$H = -a(I-\hat{I})^2 - hI - b(P-\hat{P})^2 - pP + \lambda(P-S) \qquad (6.3)$$

H is concave in P and maximum principle conditions are:

- maximizing Hamiltonian with respect to P (remains the same as for the standard formulation of the maximum principle)

$$\frac{\partial H}{\partial P} = \lambda - 2b(P-\hat{P}) - p = 0 \qquad (6.4)$$

- the equation of motion and initial condition for state variable (also remains the same)

$$\dot{I} = \frac{\partial H}{\partial \lambda} = P - S \qquad I(0) = I_0 \qquad (6.1')$$

- the equation of motion for adjoint variable and transversality condition for free endpoint problem, for current value formulation

$$\dot{\lambda} = -\frac{\partial H}{\partial I} + \rho\lambda = 2a(I-\hat{I}) + h + \rho\lambda \qquad \lambda(T) = 0 \qquad (6.5)$$

From (6.4) the decision rule for the optimal path of the control variable is

$$P^* = \hat{P} + \frac{1}{2b}(\lambda - p) \qquad (6.4')$$

Substituting (6.4') into (6.1) gives

$$\dot{I}^{*} = \hat{P} + \frac{1}{2b}(\lambda - p) - S \tag{6.6}$$

The (6.5), (6.6) and initial condition I(0)=I_0, together create two point boundary value problem (TPBVP) that will be solved as an simultaneous system of the first order differential equations

$$\dot{I} \qquad - \frac{1}{2b}\lambda = \hat{P} - \frac{p}{2b} - S \tag{6.7}$$

$$\dot{\lambda} - 2aI - \rho\lambda = h - 2a\hat{I}$$

It can be expressed in the matrix form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{I} \\ \dot{\lambda} \end{bmatrix} + \begin{bmatrix} 0 & \frac{-1}{2b} \\ -2a & -\rho \end{bmatrix}\begin{bmatrix} I \\ \lambda \end{bmatrix} = \begin{bmatrix} \hat{P} - \frac{p}{2b} - S \\ h - 2a\hat{I} \end{bmatrix} \tag{6.8}$$

or in short

$$\mathrm{I}\mathbf{u} + \mathrm{K}\mathbf{v} = \mathbf{d} \tag{6.9}$$

The solution consists of the sum of the complementary function (for **d**=**0**) and the particular solutions.

For finding complementary functions, the trial solutions for **v** is used

$$\mathbf{v} = \begin{bmatrix} I \\ \lambda \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix} e^{rt}$$

which implies

$$\mathbf{u} = \begin{bmatrix} \dot{I} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix} re^{rt} = r\mathbf{v}$$

Substituting **v** and **u** into reduced equation

$$\mathrm{I}\mathbf{u} + \mathrm{K}\mathbf{v} = \mathbf{0} \tag{6.10}$$

gives:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix} re^{rt} + \begin{bmatrix} 0 & \frac{-1}{2b} \\ -2a & -\rho \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix} e^{rt} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(rI + K\right)\begin{bmatrix} m \\ n \end{bmatrix} e^{rt} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (6.10')$$

Multiplying (6.10`) with the e^{-rt} gives the system of characteristic equations

$$(rI + K)\begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (6.11)$$

which has nontrivial solution only under condition det(rI+K)=0 from which the equation for characteristic routs is deduced

$$\det\left(\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{2b} \\ -2a & -\rho \end{bmatrix}\right) = 0$$

$$\begin{vmatrix} r & -\frac{1}{2b} \\ -2a & r-\rho \end{vmatrix} = 0$$

$$r^2 - \rho r - \frac{a}{b} = 0$$

Finally, the solutions of characteristic routs are

$$r_{2,1} = \frac{\rho \pm \sqrt{\rho^2 + \frac{4a}{b}}}{2} \qquad (6.12)$$

It is obvious that r_1 and r_2 are two different real numbers because a and b are assumed positive and consequently the expression $\rho^2 + \frac{4a}{b}$ is positive. Also, r_2 is obviously positive and r_1 is negative for the same reason for which they are real:

$$\rho - \sqrt{\rho^2 + \frac{4a}{b}} < 0$$

$$\rho < \sqrt{\rho^2 + \frac{4a}{b}}$$

Since $\rho > 0$ and $\rho^2 + \frac{4a}{b} > 0$, squaring the inequation gives

$$\rho^2 < \rho^2 + \frac{4a}{b}$$

which is true. Introducing the r_1 and r_2 routs into (6.11) it becomes

$$\begin{bmatrix} r_{1,2} & -\frac{1}{2b} \\ -2a & r_{1,2} - \rho \end{bmatrix} \begin{bmatrix} m_{1,2} \\ n_{1,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (6.13)$$

Since the rows of this matrix and hence the equations from this system are linearly dependent the relation between m and n can be expressed from any of them. For example, from the first one and for the first rout, it gives

$$n_1=2br_1m_1 \quad (6.14)$$

and similarly for the second

$$n_2=2br_2m_2 \quad (6.15)$$

Introducing the unknown constants A_1 and A_2 gives the following equations

$$m_1=A_1 \qquad m_2=A_2$$
$$n_1=2br_1A_1 \qquad n_2=2br_2A_2$$

and complementary solution can be expressed as:

$$\begin{bmatrix} I_c \\ \lambda_c \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{2} m_i e^{r_i t} \\ \sum_{i=1}^{2} n_i e^{r_i t} \end{bmatrix} = \begin{bmatrix} A_1 e^{r_1 t} + A_2 e^{r_2 t} \\ 2br_1 A_1 e^{r_1 t} + 2br_2 A_2 e^{r_2 t} \end{bmatrix} \quad (6.16)$$

If the particular solution for I(t) is given with the function named D

$$\bar{I}(t) = D(t) \quad (6.17)$$

from that follows

$$\dot{\bar{I}}(y) = \dot{D}(t)$$

Substituting it into the first equation of the system (6.7) gives the equation:

$$\dot{D} - \frac{1}{2b}\bar{\lambda} = \hat{P} - \frac{p}{2b} - S(t)$$

from which the expression for the particular solution of adjoint variable λ can be deduced

$$\bar{\lambda} = p + 2b(S(t) + \dot{D}(t) - \hat{P}) \qquad (6.18)$$

and finally the entire solution for state and adjoint variable can be expressed as

$$\begin{bmatrix} I \\ \lambda \end{bmatrix} = \begin{bmatrix} A_1 e^{r_1 t} + A_2 e^{r_2 t} + D(t) \\ 2b(r_1 A_1 e^{r_1 t} + r_2 A_2 e^{r_2 t} - \hat{P} + S(t) + \dot{D}(t)) + p \end{bmatrix} \qquad (6.18')$$

For finding constants A_1 and A_2 the initial condition $I(0)=I_0$ and the transversallity condition $\lambda(T)=0$ should be introduced into (6.16). It gives the equation

$$\begin{bmatrix} I_0 \\ 0 \end{bmatrix} = \begin{bmatrix} A_1 + A_2 + D(0) \\ 2b(r_1 A_1 e^{r_1 T} + r_2 A_2 e^{r_2 T} + \dot{D}(T) + S(T) - \hat{P}) + p \end{bmatrix} \qquad (6.19)$$

that can be transformed into the system

$$\begin{aligned} A_1 + \qquad A_2 &= I_0 - D(0) \\ r_1 e^{r_1 T} A_1 + r_2 e^{r_2 T} A_2 &= \hat{P} - S(T) - \dot{D}(T) - \frac{p}{2b} \end{aligned} \qquad (6.20)$$

or, if the constants d_1 and d_2

$$\begin{aligned} d_1 &= I_0 - D(0) \\ d_2 &= \hat{P} - S(T) - \dot{D}(T) - \frac{p}{2b} \end{aligned} \qquad (6.20')$$

are introduced, the system (6.20) becomes

$$\begin{aligned} A_1 + \qquad A_2 &= d_1 \\ r_1 e^{r_1 T} A_1 + r_2 e^{r_2 T} A_2 &= d_2 \end{aligned} \qquad (6.20'')$$

and it can be expressed in the simpler, matrix form:

$$\begin{bmatrix} 1 & 1 \\ r_1 e^{r_1 T} & r_2 e^{r_2 T} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

The determinant of the system is

$$Det = r_2 e^{r_2 T} - r_1 e^{r_1 T}$$

and the solution for constant A_1 and A_2 are

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \frac{1}{r_2 e^{r_2 T} - r_1 e^{r_1 T}} \begin{bmatrix} r_2 e^{r_2 T} & -1 \\ -r_1 e^{r_1 T} & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

or

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} \dfrac{d_1 r_2 e^{r_2 T} - d_2}{r_2 e^{r_2 T} - r_1 e^{r_1 T}} \\ \dfrac{d_2 - d_1 r_1 e^{r_1 T}}{r_2 e^{r_2 T} - r_1 e^{r_1 T}} \end{bmatrix} \qquad (6.21)$$

(remember that $r_1<0$ and $r_2>0$)

Now, using (6.18\`), (6.4\`), constants A_1 and A_2 given with (6.21), constants d_1 and d_2 introduced with (6.20\`) and routs r_1 and r_2 from (6.12), the expressions for optimal paths of the state variable I, control variable P and adjoint variable λ are given with following equations:

$$\begin{aligned} I^* &= A_1 e^{r_1 t} + A_2 e^{r_2 t} + D(t) \\ P^* &= r_1 A_1 e^{r_1 t} + r_2 A_2 e^{r_2 t} + S(t) + \dot{D}(t) \\ \lambda^* &= 2b(r_1 A_1 e^{r_1 t} + r_2 A_2 e^{r_2 t} - \hat{P} + S(t) + \dot{D}(t)) + p \end{aligned} \qquad (6.22)$$

Hamiltonian is concave in control variable P.

Proof:

Partial derivation of Hamiltonian (6.3) with respect to P twice gives:

$$\frac{\partial H}{\partial P} = -2b(P - \hat{P}) - p$$

$$\frac{\partial^2 H}{\partial P^2} = -2b$$

b is assumed positive constant and it proves that Hamiltonian is concave in P. Since Hamiltonian is concave in control variable, the necessary conditions are also sufficient for maximizing it.

6.1.2 Examples and analysis of a special case with polynomial demand and zero continuous interest rate

For a special case, described in heading of the chapter, some numerical examples will be solved and their graphs will be drawn in order to analyze and visualize the sensitivity of optimal paths on different parameters of the model. The planning horizon will be a finite and not very large number so that a zero value for the interest rate can be assumed. The demand will be in the form of n^{th} degree polynomial.

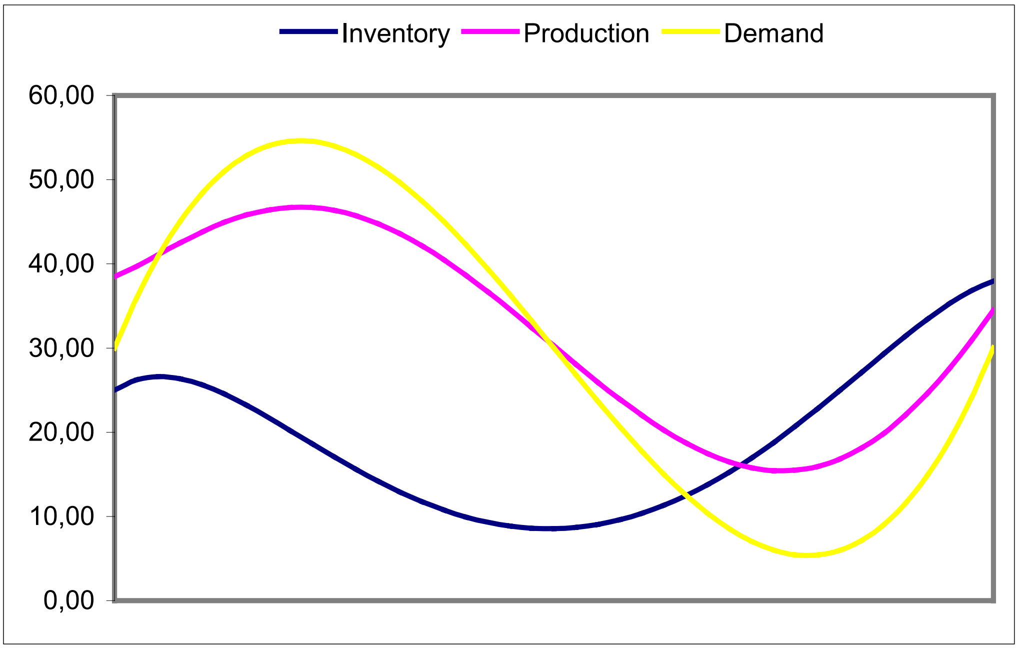
$$S(t) = C_n t^n + C_{n-1} t^{n-1} + \quad \ldots \quad + C_1 t + C_0$$

Then it is easy to show that the particular solution for I(t), $\bar{I}(t) = D(t)$ has a form:

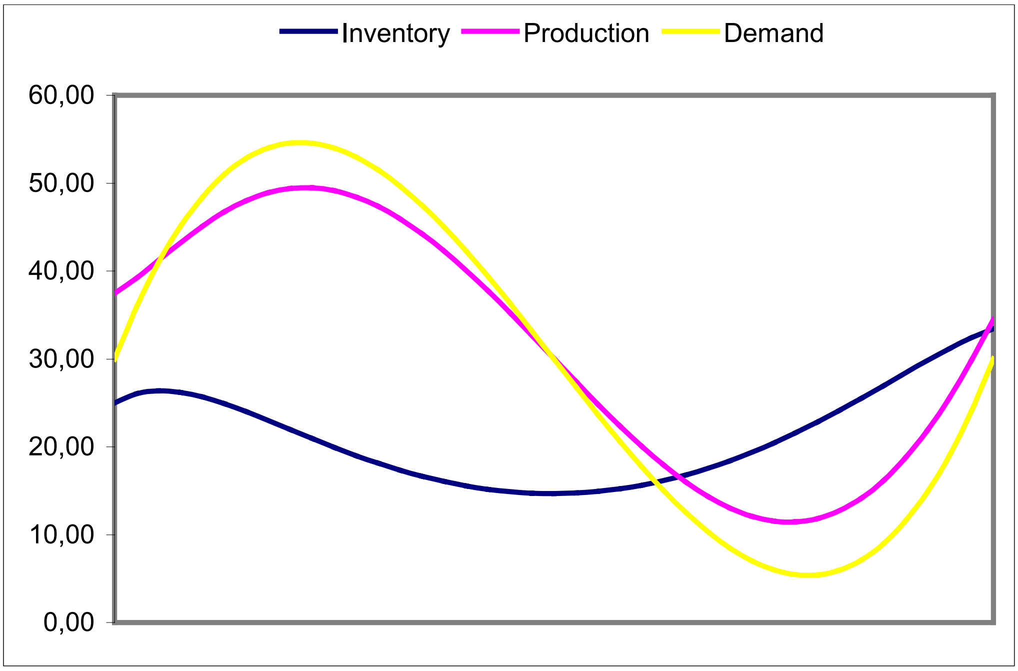
$$D(t) = \hat{I} + \sum_{p=1}^{\mathrm{int}(\frac{n+1}{2})} \frac{1}{\alpha^{2p}} S^{(2p-1)} - \frac{h}{2a}$$

1) $S(t) = t^3 - 12t^2 + 32t + 30$

$\hat{P} = 35$ $\hat{I} = 20$ $\rho = 0$ $a = 1$ $b = 1$ $h = 1$ $p = 1$ $I_0 = 25$ $T = 8$



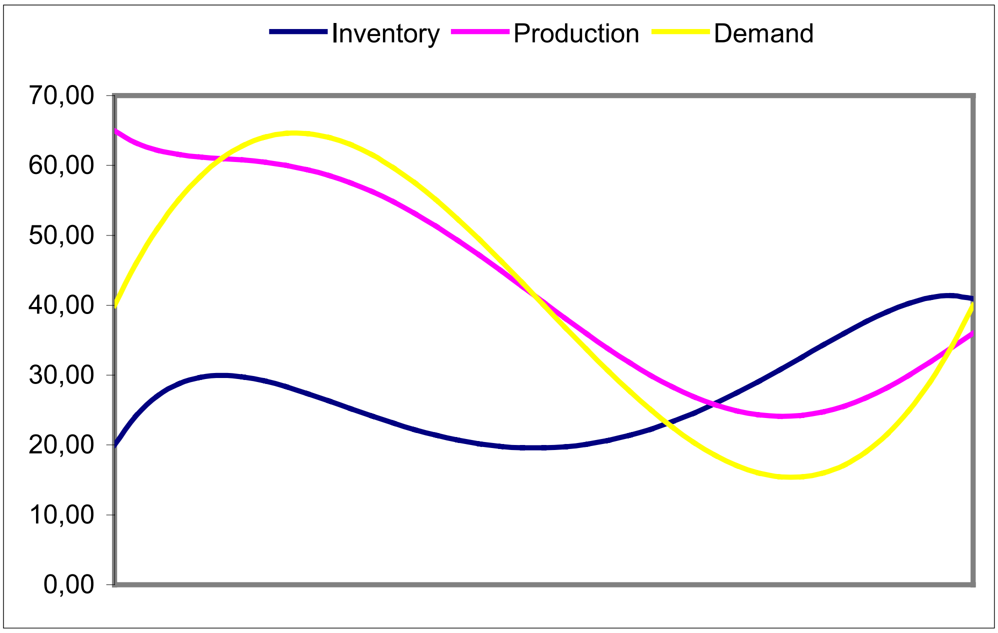
a=2 and the other parameters of the model remain the same



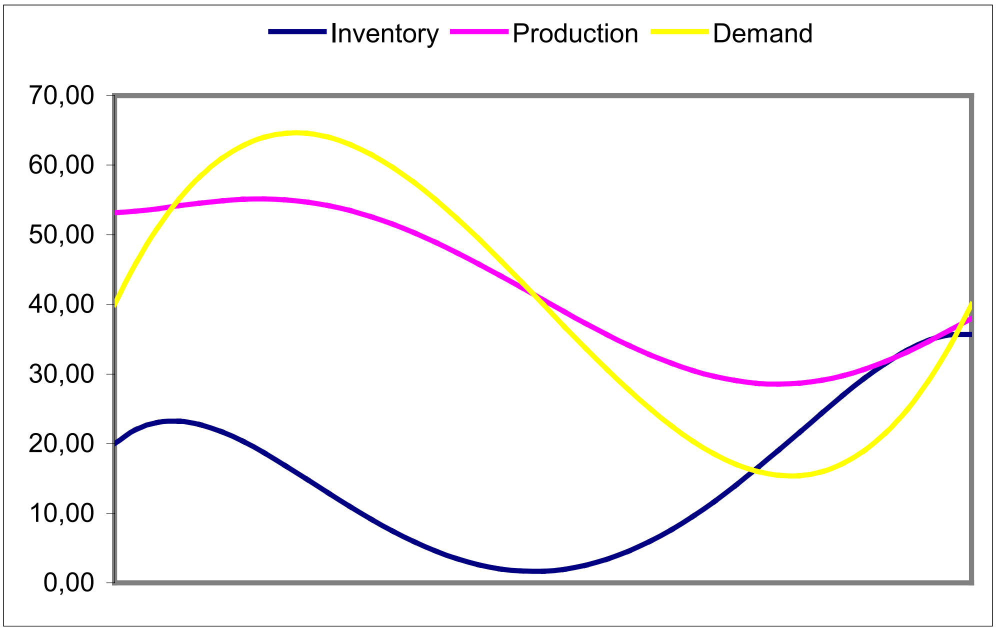
It can be concluded from this two graphs that, when the costs of deviation from optimal inventory increase, the optimal path for inventory is getting smoother and closer to desired inventory, whose value is 20 in this case. At the end of planning period, inventory is becoming smaller. At the same time, the optimal path for production is becoming less smooth.

2) $S(t) = t^3 - 12t^2 + 32t + 40$

$\hat{P} = 40$ $\hat{I} = 35$ $\rho=0$ a= 0,5 b=0,5 h=4 p=4 $I_0=20$ T=8



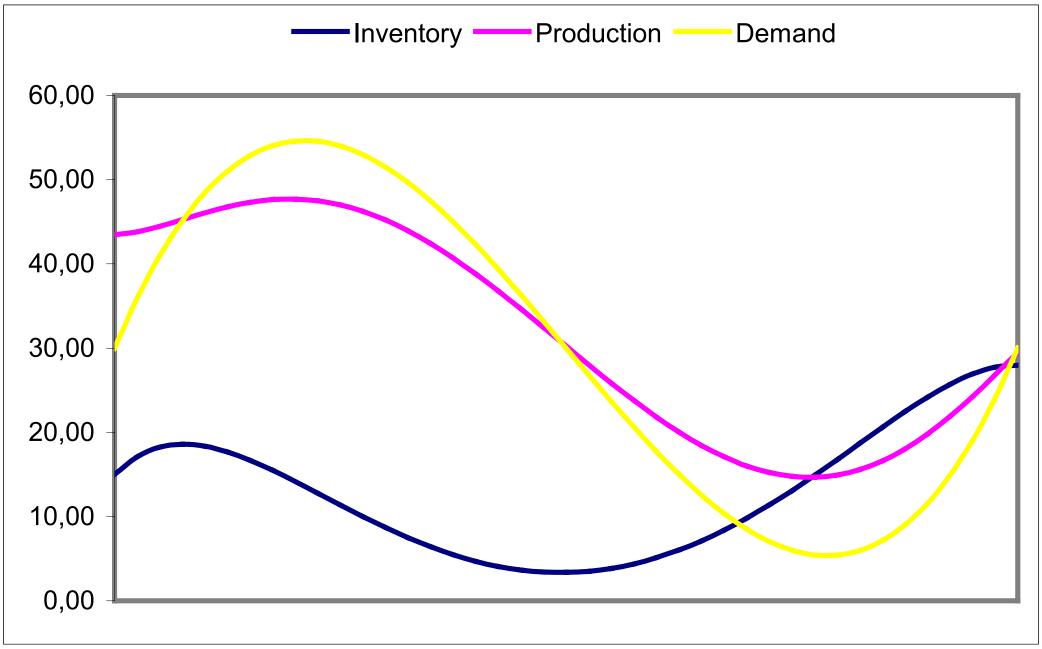
b=1 and the other parameters of the model remain the same



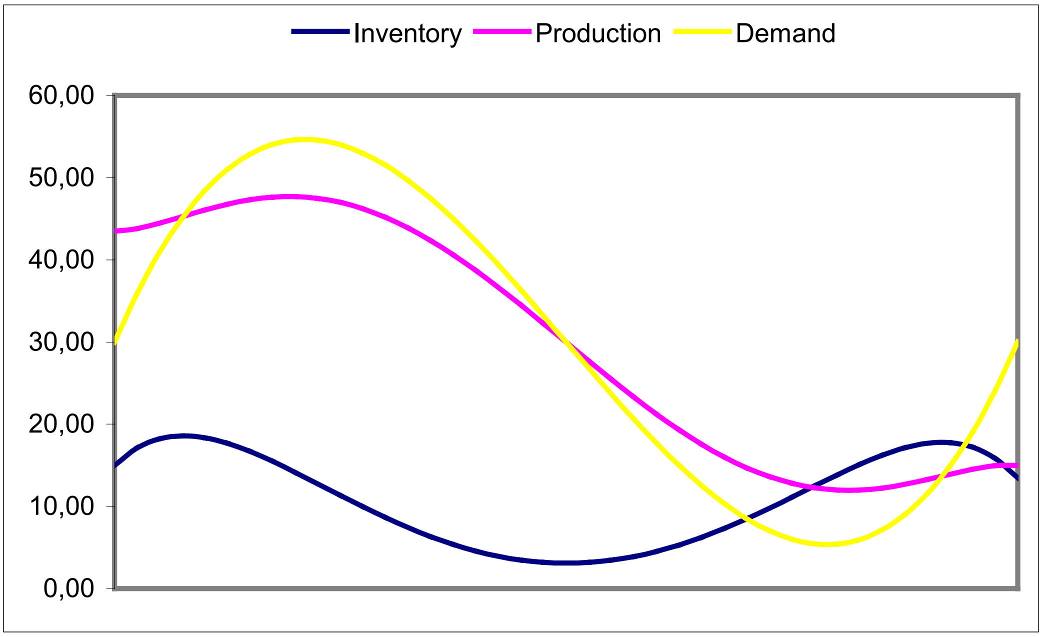
When the costs of deviation from optimal production increase, the optimal production rate is becoming smoother and closer to desired rate, whose value is 40. It could be also noticed that the optimal inventory level is becoming less smooth and lower most of the time and in the end of planning period.

3) $S(t) = t^3 - 12t^2 + 32t + 30$

$\hat{P} = 30$ $\hat{I} = 15$ $\rho=0$ a= 1 b=1 h=1 p=1 $I_0=15$ T=8



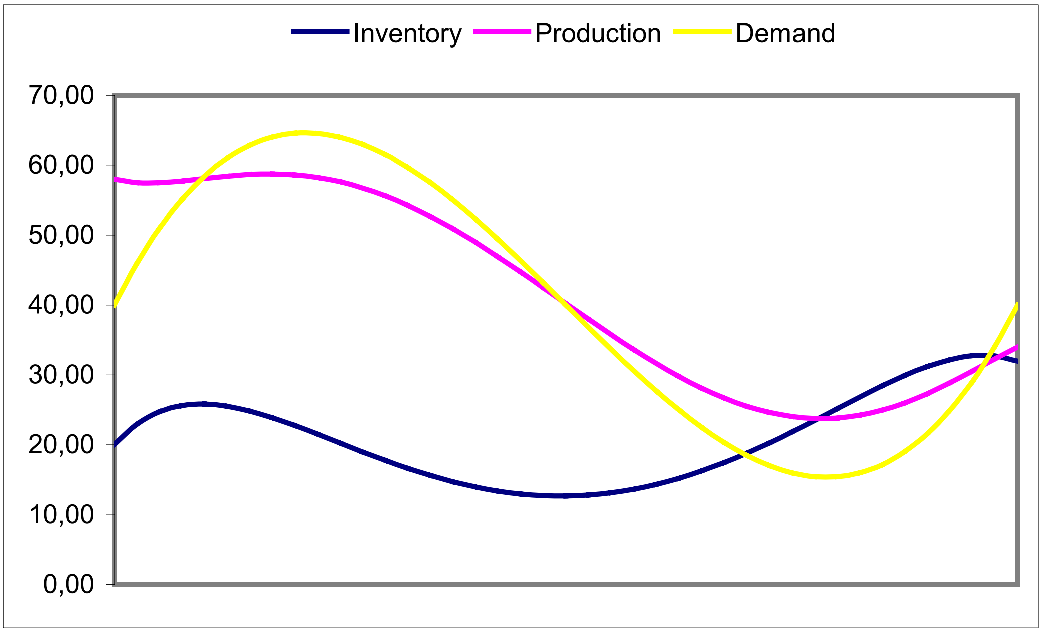
p=30 and the other parameters of the model remain the same



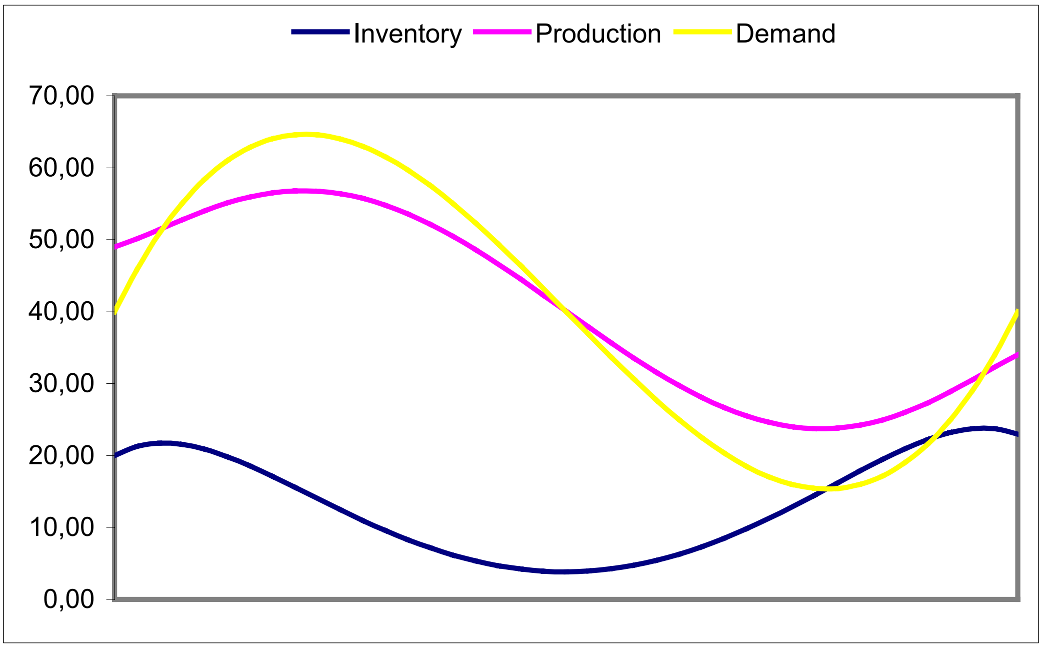
When the linear costs of production increase, in the beginning of planning period, it does not affect any of the optimal path values considerably. However, later both optimal path values for inventory and production are becoming lower. The production, due to increase of respective linear costs, and the inventory, because of the lower production with same demand.

4) $S(t) = t^3 - 12t^2 + 32t + 40$

$\hat{P} = 35$ $\hat{I} = 25$ $\rho=0$ $a= 0,5$ $b=0,5$ $h=1$ $p=1$ $I_0=20$ $T=8$



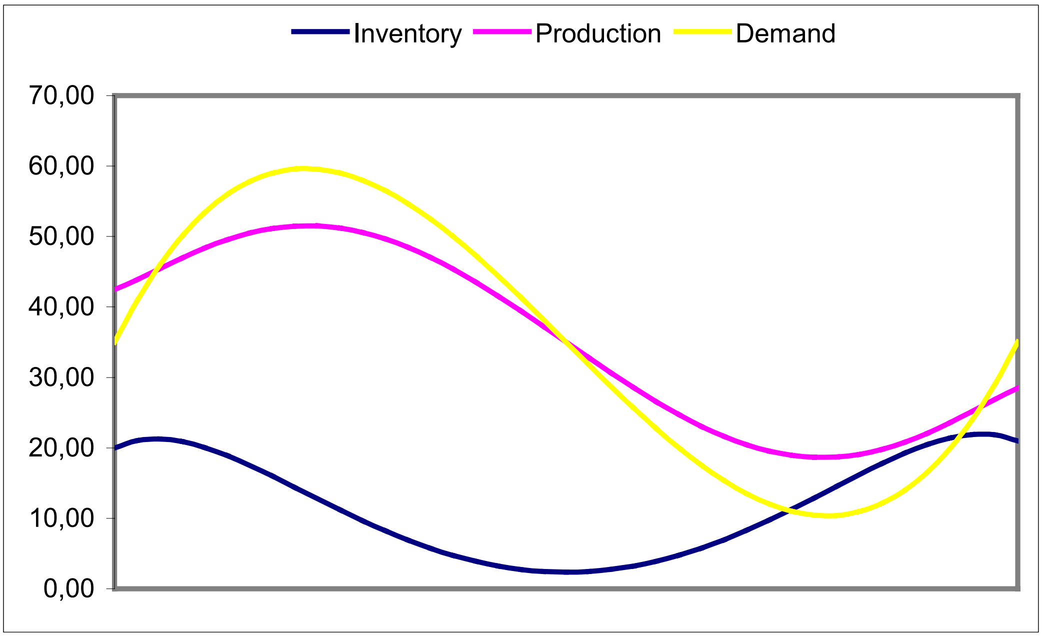
h=10 and the other parameters of the model remain the same



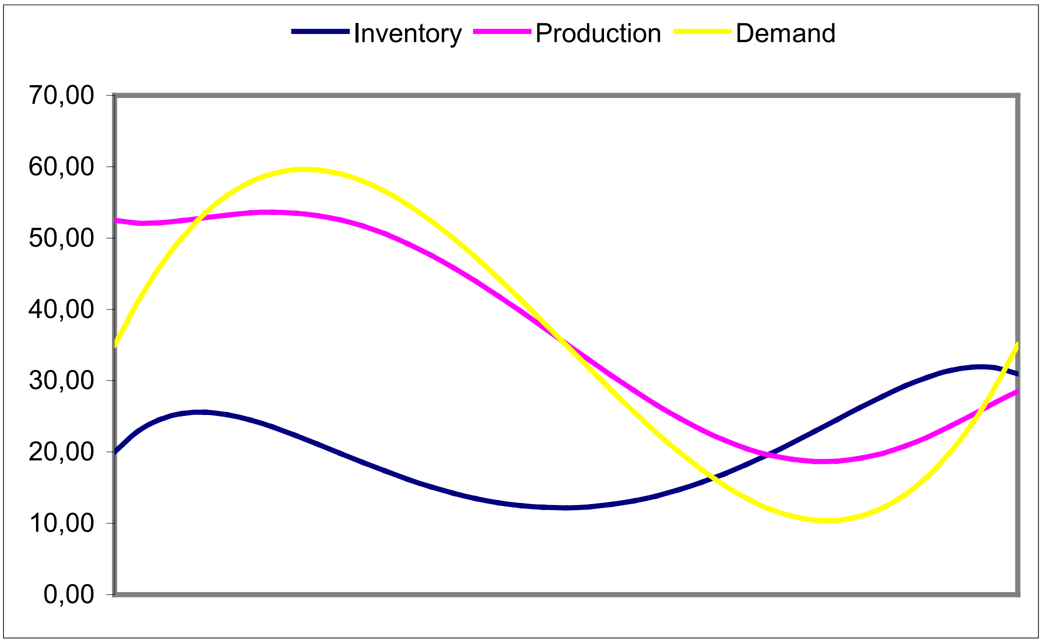
When the linear cost of keeping inventory increases, the inventory level decreases and the optimal production rate changes only in the beginning of planning period. This is because the lower production in the beginning decreases the inventory level, and later, with production as in above example, the inventory remains equally lower till the end of planning horizon.

5) $S(t) = t^3 - 12t^2 + 32t + 35$

$\hat{P} = 30$ $\hat{I} = 15$ $\rho = 0$ $a = 1$ $b = 1$ $h = 3$ $p = 3$ $I_0 = 20$ $T = 8$



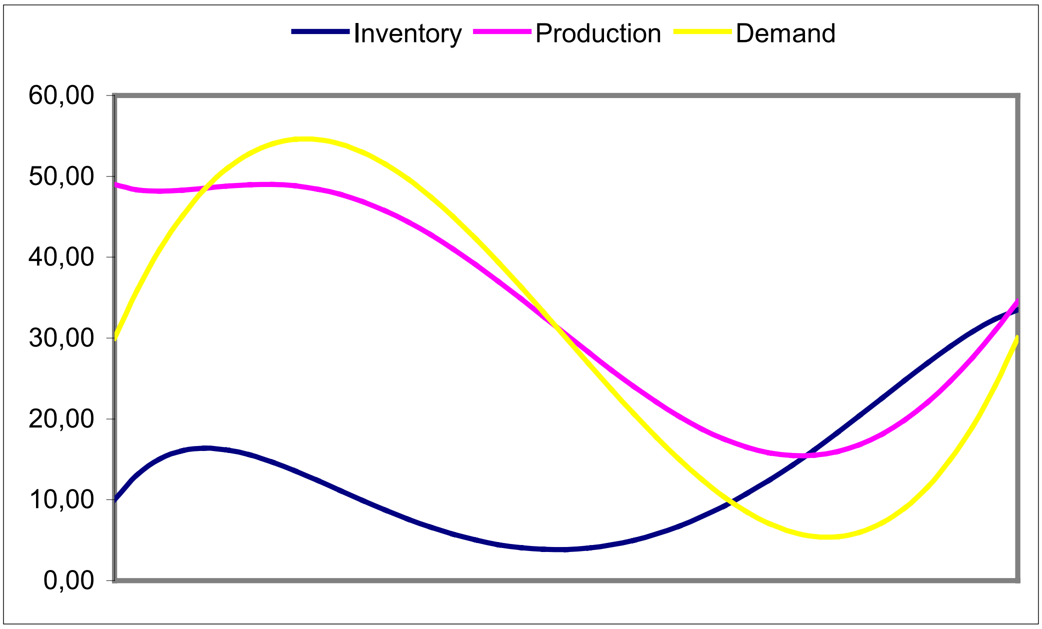
$\hat{I} = 25$ and the other parameters of the model remain the same



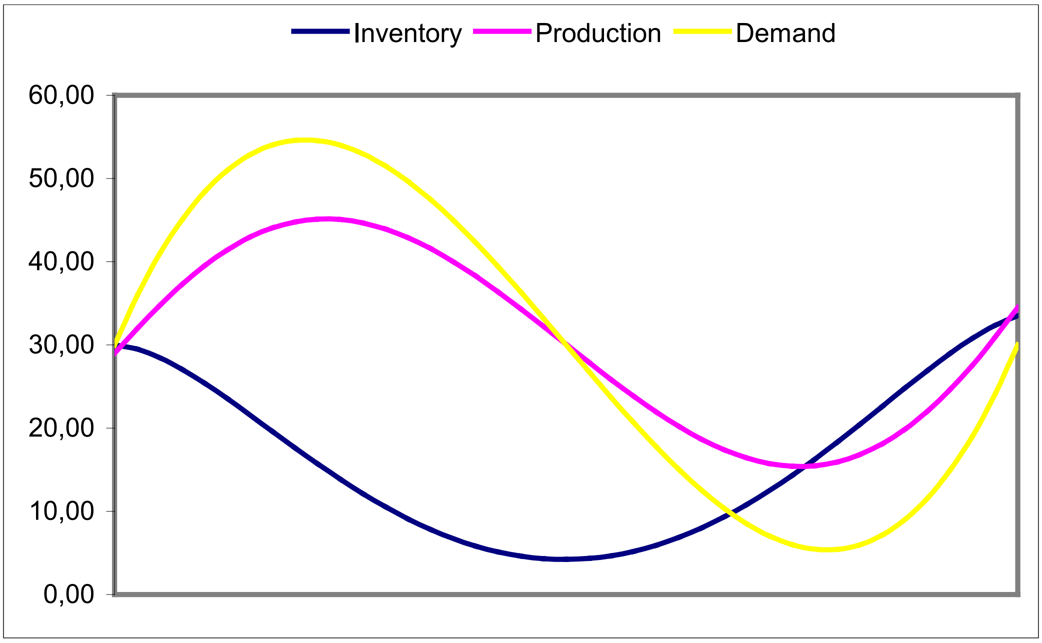
When the desired inventory level increases, the production rate changes only in the beginning of the planning horizon to increase the inventory level closer to the desired. After that there is no need for higher production any more and it remains at the same level as on the previous graph, to keep the inventory level with the same shape but equally higher as compared to the previous figure until the end of planning horizon.

6) $S(t) = t^3 - 12t^2 + 32t + 30$

$\hat{P} = 35$ $\hat{I} = 20$ $\rho=0$ $a= 1$ $b=1$ $h=10$ $p=1$ $I_0=10$ $T=8$



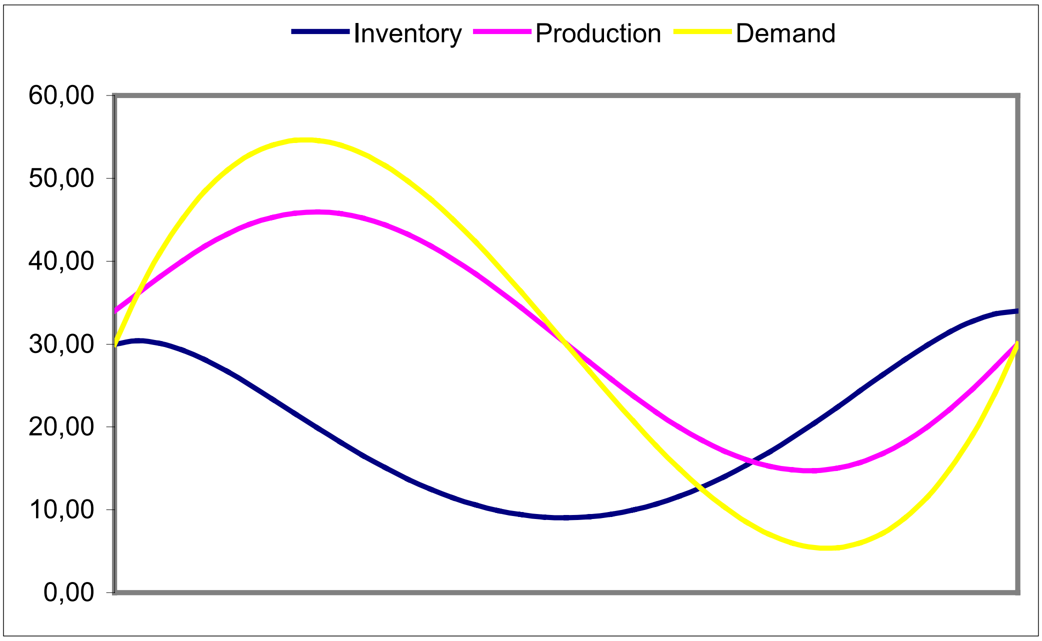
$I_0=30$ and the other parameters of the model remain the same



Increase of initial inventory has an influence on the optimal paths only in the beginning of the planning period but very soon it becomes irrelevant.

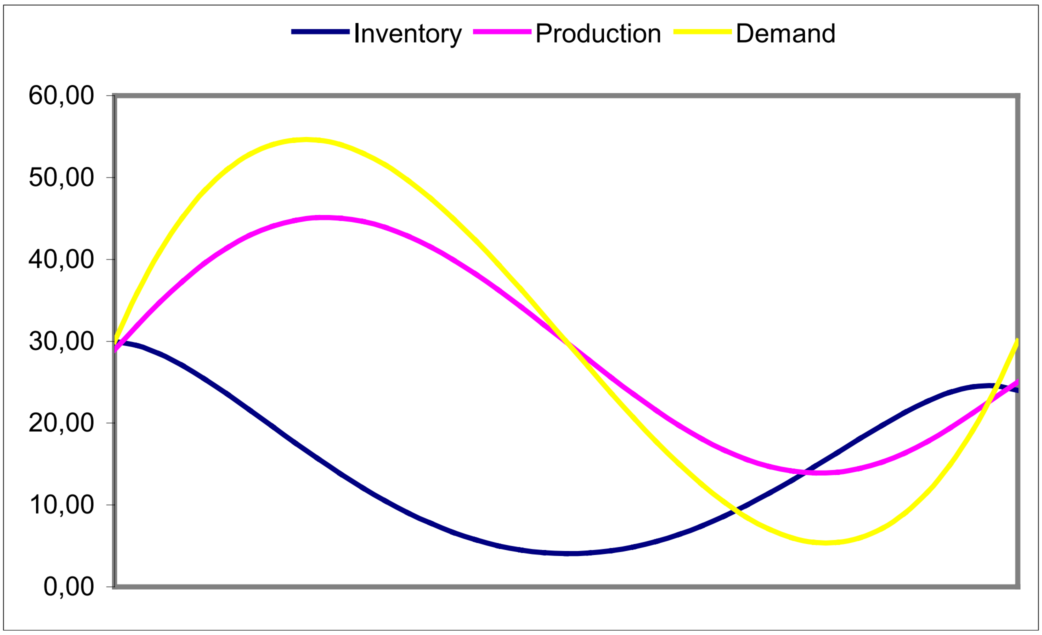
7) a) $S(t) = t^3 - 12t^2 + 32t + 30$

$\hat{P} = 30$ $\hat{I} = 20$ $\rho = 0$ $a = 1$ $b = 1$ $h = 0$ $p = 0$ $I_0 = 30$ $T = 8$



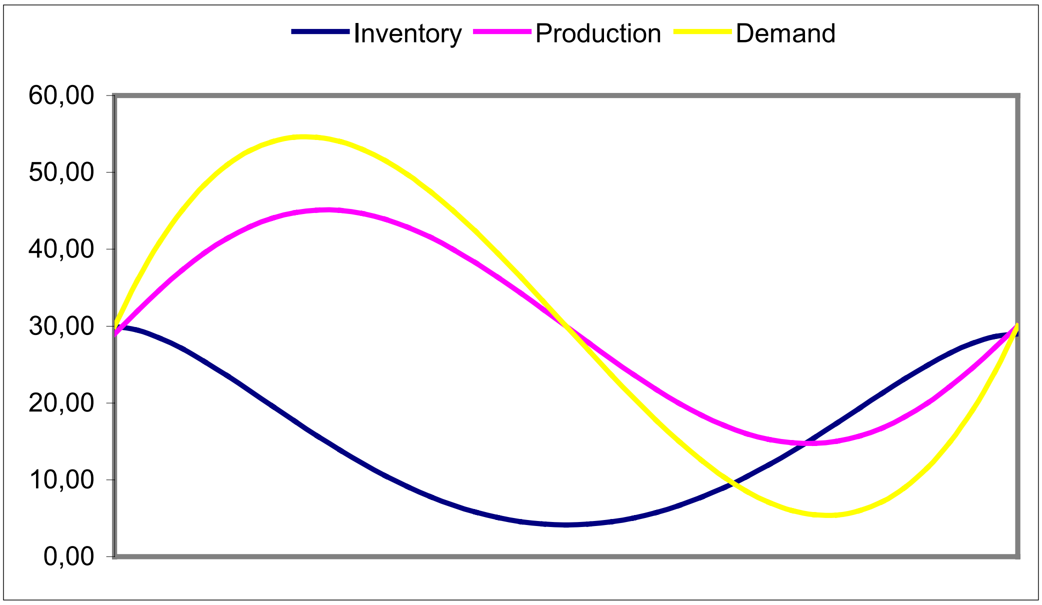
In this example, there are no linear costs but only costs for deviations of the goal levels, which means that it represents a usual HMMS quadratic type of model.

b) h=10 p=10 and the other parameters of the model remain the same



Introducing both linear costs causes decrease of production rate at the beginning and in the end but not all the time. Decrease of production in the beginning causes decrease of inventory level whose path then remains equally lower all the time, until it comes near the end of planning period. Then it goes down again because of the repeated decrease of production.

c) h=10 p=0 and the other parameters of the model remain the same



Comparing cases a) and c), which means introducing only linear cost for holding inventories, causes lowering of production only in the beginning of the planning horizon, which lowers also the inventory level in the beginning and then it further remains equally lower all the time within the period.

The global conclusion resulting from the examined examples is that these two kinds of costs influence the optimal solutions in different ways. The increase of quadratic costs (the deviations costs) affects more the shape of optimal production and inventory paths causing their smoothing, while on the other side, the increase of linear costs, lowers the level of the paths.

It proves the idea (hypotheses) that it is important to introduce linear costs and to distinguish these two types of costs.

6.2 Application of the basic model for the case with infinite planning horizon

When time horizon is long or even infinite, it is important to discount because otherwise all the solutions would be unbounded and would give an infinite values (which, in our case, is the worst, since we are considering costs and aim to minimize them). So the continuous discount rate ρ in this case will be assumed ρ>0. The constants A_1 and A_2 from the model become

$$\lim_{T\to\infty} A_1 = \lim_{T\to\infty} \frac{d_1 r_2 e^{r_2 T} - d_2}{r_2 e^{r_2 T} - r_1 e^{r_1 T}}$$

Dividing both numerator and denominator by continuous function $e^{r_2 t}$ gives

$$\lim_{T\to\infty} A_1 = \lim_{T\to\infty} \frac{d_1 r_2 + \dfrac{d_2}{e^{r_2 T}}}{r_2 - r_1 e^{(r_1 - r_2)T}}$$

Since $r_1<0$, $r_2>0$, which implies $r_1 - r_2 < 0$, it follows that

$$\lim_{T\to\infty} A_1 = d_1 \qquad (6.2.1)$$

and

$$\lim_{T\to\infty} A_2 = \lim_{T\to\infty} \frac{d_2 - d_1 r_1 e^{r_1 T}}{r_2 e^{r_2 T} - r_1 e^{r_1 T}} = 0 \qquad (6.2.2)$$

So, in this case from (6.22), (6.2.1) and (6.2.2) the optimal paths become:

$$\begin{aligned} I^*(t) &= d_1 e^{r_1 t} + D(t) \\ P^*(t) &= r_1 d_1 e^{r_1 t} + S(t) + \dot{D}(t) \\ \lambda^*(t) &= 2b\left[r_1 d_1 e^{r_1 t} - \hat{P} + S(t) + \dot{D}(t) \right] + p \end{aligned} \qquad (6.2.3)$$

Since $r_1<0$, which implies $d_1 e^{r_1 t}$, converges to zero when t tends to infinity, it can be seen that $I^*(t)$ converges to its particular solution D(t) which is actually an intermediate equilibrium level. It means that optimal "time path" converges and fulfills condition for dynamic stability of equilibrium.

6.2.1 Specialization of the model for the case with constant positive demand

For constant S, particular solution for $I^*(t)$ becomes constant $\bar{I}(t) = D$ and the particular solution for λ is also constant given by $\bar{\lambda} = p + 2b(S - \hat{P})$. $\dot{\bar{I}} = 0$ and $\dot{\bar{\lambda}} = 0$. When these are introduced into the system of differential equations (6.8), it changes into the following matrix equation:

$$\begin{bmatrix} 0 & -\dfrac{1}{2b} \\ -2a & -\rho \end{bmatrix} \begin{bmatrix} \bar{I} \\ \bar{\lambda} \end{bmatrix} = \begin{bmatrix} \hat{P} - \dfrac{p}{2b} - S \\ h - 2a\hat{I} \end{bmatrix}$$

The determinant of the matrix on the left side is det=-a/b and the solutions for $\bar{I}$ and $\bar{\lambda}$ are obtained as:

$$\begin{bmatrix} \bar{I} \\ \bar{\lambda} \end{bmatrix} = -\frac{b}{a} \begin{bmatrix} -\rho & \dfrac{1}{2b} \\ 2a & 0 \end{bmatrix} \begin{bmatrix} \hat{P} - \dfrac{p}{2b} - S \\ h - 2a\hat{I} \end{bmatrix}$$

or

$$\begin{bmatrix} \bar{I} \\ \bar{\lambda} \end{bmatrix} = \begin{bmatrix} \hat{I} + \dfrac{\rho b}{a}(\hat{P} - S) - \dfrac{\rho p + h}{2a} \\ 2b(S - \hat{P}) + p \end{bmatrix} \tag{6.2.4}$$

So

$$D = \hat{I} + \frac{\rho b}{a}(\hat{P} - S) - \frac{\rho p + h}{2a} \tag{6.2.5}$$

From the definition of constants d_1, d_2 in (6.20`), since D is constant and
$\dot{D}=0$, they became:

$$d_1 = I_0 - D$$

$$d_2 = \hat{P} - S - \frac{p}{2b} \tag{6.2.6}$$

or

$$d_1 = I_0 - \hat{I} - \frac{\rho b}{a}(\hat{P} - S) + \frac{\rho p + h}{2a}$$

$$d_2 = \hat{P} - S - \frac{p}{2b} \tag{6.2.6'}$$

The optimal paths from (6.2.3) and with constant S (and constant D) are

$$I^* = d_1 e^{r_1 t} + D$$

$$P^* = r_1 d_1 e^{r_1 t} + S \tag{6.2.7}$$

$$\lambda^* = 2b(r_1 d_1 e^{r_1 t} - \hat{P} + S) + p$$

6.2.2 Extension of the previous model by introducing constraint on the control variable

Until now, I assumed that $\hat{P}$ was sufficiently large and I_0 sufficiently small so that P will never become zero. It means that I included the interior solution implicitly, which hypothesis insufficiently reflects reality.

Now, I will consider the case where there is constraint on the control variable P requiring it to be nonnegative. ($P(t) \geq 0$). I will assume again that demand S is a positive constant and continuous discount rate ρ is positive. Since the solution now can be boundary, different optimal decision rule for production given by following equation will be used

$$P^* = \max\{r_1 d_1 e^{r_1 t} + S, 0\} \tag{6.2.8}$$

The first possibility is for P interior and the second for P on its boundary. In the first possibility, as I have shown before, the optimal paths for interior solution for all three variables are given with (6.2.7).

6.2.3 Analysis of solutions depending on initial condition for inventory with examples

In this chapter, I will perform some analysis as to how the behavior of the model and optimal paths of both production and inventory depend on the level of initial inventory.

Case 1

If I_0=D (remember $\bar{I}(t) = D$) from (6.2.6) follows that d_1=0 and from (6.2.7) that P^*=S which is positive so the solution is interior and $I^* = D$

From (6.4') follows

$$\bar{P} = \hat{P} + \frac{1}{2b}(\bar{\lambda} - p)$$

and from (6.18)

$$\bar{\lambda} = p + 2b(S - \hat{P})$$

It can be deduced from these equations that $\bar{P} = S$. It means that in this case, since $I_0 = \bar{I}$ (or D) the optimal production path is $P^* = \bar{P}$ for every t.

The conclusion is:

If the initial inventory equals particular solution for inventory, then the solution of optimal production equals to its particular solution and they both equal to demand.

$$P^* = \bar{P} = S$$

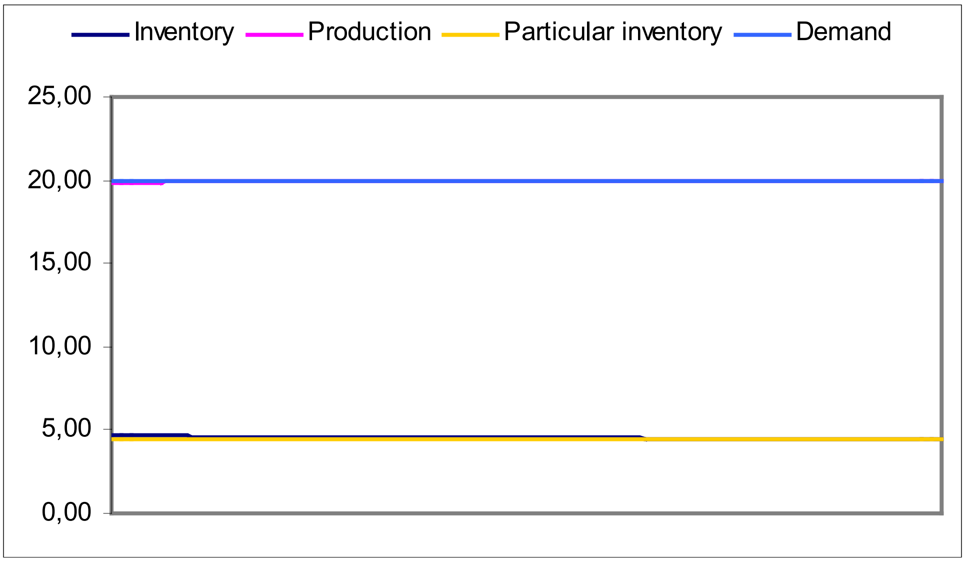
It is interesting to notice that in this situation the optimal path for production depends only on demand and not on the parameters of the model.

Of course, if any parameter in the model is changed, then the particular solution D for the inventory changes, and if the manager wants to keep production equal to demand, he must change initial inventory, setting it to the value of particular solution D.

8.) Example

$\hat{P}$ =25 $\hat{I}$=15 ρ=0,5 S=20 a=0,5 b=0,5 h=8 p=10

⇒ D=4,5 r_1= -0,780776 I_0=4,7 (D=4,5)



This example illustrates the conclusions given above. For initial inventory I did not take exactly the same value of particular optimal solution for inventory, but slightly higher, to make the optimal production and inventory paths visible. (If the values were the same, the demand and particular inventory paths would "cover" the optimal production and inventory paths respectively.)

Case 2

For $I_0 \neq D$, from (6.2.6), (6.2.8) and (6.4') the optimal solution is given by

$$P^*(t) = \max\{r_1(I_0 - D)e^{r_1 t} + S, 0\} \qquad (6.2.9)$$

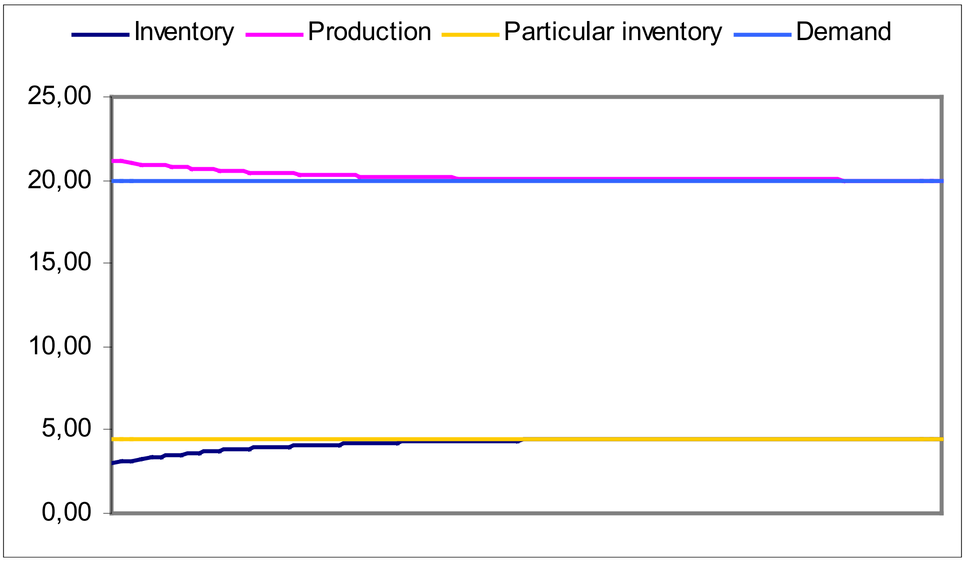
$$= \max\{\hat{P} + \frac{1}{2b}(\lambda - p), 0\}$$

Case 2.1

For $I_0 \leq D$ (since $r_1<0$ and $(I_0-D)\leq 0$) follows that optimal production is always nonnegative, meaning that solution is interior given with (6.2.7).

9) Example

$I_0=3<D=4,5$ and the other parameters of the model remain the same



This example illustrates how the paths are moving when initial inventory level is lower than the value of particular optimal solution for inventory. At the beginning, its level is below the particular inventory line and consequently the production path is higher than demand. But as production path approaches the demand path, the inventory level tends to its particular solution (meaning that it has the property of dynamic stability).

Case 2.2

For $I_0>D$, since $r_1(I_0-D)$ is negative, $e^{r_1 t}$ is decreasing, S is assumed constant, and it follows from (6.2.9) that P^* is increasing. So if the value for the zero moment P(0) is positive, optimal production solution $P^*(t)$ will be positive, all the time. I will now find the initial conditions for which this is true.

The value of the initial production is $P(0)=r_1(I_0-D)+S$ and if it must be positive:

$$r_1(I_0-D)+S>0 \qquad /:r_1 \quad (r_1<0)$$

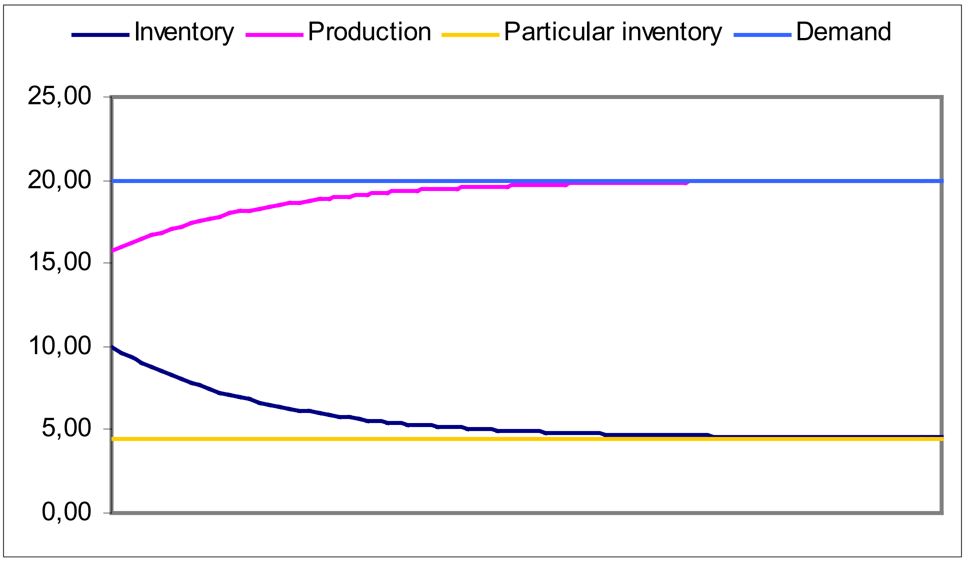
$$I_0 \langle D-\frac{S}{r_1}$$

The conclusion is:

If the inventory level is lower then the value of $D-\frac{S}{r_1}$, the value of P(0) is positive and consequently $P^*(t)$ is positive and interior.

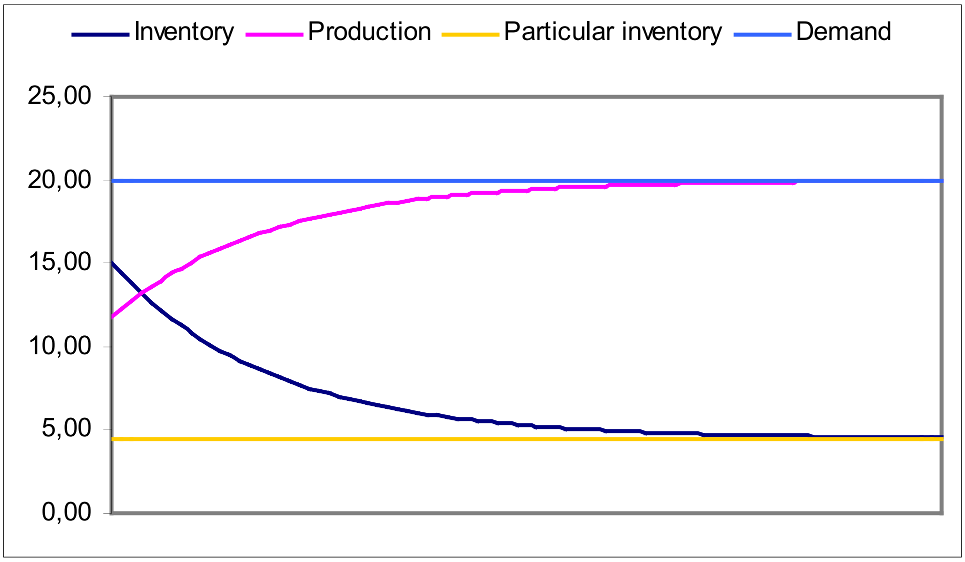
10) Example

$D=4{,}5<I_0=10<30{,}115528$ ($I_0<\hat{I}=15$) and the other parameters of the model remain the same $\Rightarrow$ $D-\frac{S}{r_1}=30{,}115528$



11) Example

$D=4{,}5<I_0=15<30{,}115528$ ($I_0=\hat{I}$) and the other parameters of the model remain the same



These two examples show the cases where the value of initial inventory is higher than value of particular optimal solution for inventory. In such situations, the production in the beginning of planning horizon is lower because there is no need for so large inventory. During the time though, production slightly increases to approach the demand and consequently the inventory level decreases and tends to its particular solution.

In all of the examples of chapter 6.2, analyzed till now, it can be noticed that they had an interior solution for production (control variable) and that as T tends to infinity, the optimal paths for production and inventory converge to their particular optimal solutions respectively.

Case 2.3

$$I_0 > D - \frac{S}{r_1} \tag{6.2.10}$$

When (6.2.10) is valid, the value of P(0) would be negative and the optimal production P^* given in (6.2.9) is zero until the moment t_1, where

$$P(t_1) = r_1(I_0 - D)e^{r_1 t_1} + S = 0$$

which implies

$$e^{r_1 t_1} = \frac{S}{r_1(D - I_0)} \tag{6.2.11}$$

What is the value of the moment t_1? From (6.2.7), (6.2.6) and (6.2.11) it can be deduced that optimal inventory in the moment t_1 would be

$$I^*(t_1) = (I_0 - D)e^{r_1 t_1} + D$$

$$= (I_0 - D)\frac{S}{r_1(D - I_0)} + D$$

$$I^*(t_1) = D - \frac{S}{r_1} \tag{6.2.12}$$

Also for $t \le t_1$ (since P^* =0) the equation of motion for inventory is different and has the following expression:

$$\dot{I} = -S$$

Solving it gives:

$$I(t)=I_0-St \tag{6.2.13}$$

The expression (6.2.13) means that, when there is no production, the inventory is decreasing from the initial inventory I_0, as the demand is spending it during the time.
Since it is valid for $t \le t_1$, the inventory for the moment t_1 is given by

$$I(t_1)=I_0-St_1 \tag{6.2.14}$$

Equating (6.2.14) with (6.2.12) gives:

$$I_0 - St_1 = D - \frac{S}{r_1}$$

$$St_1 = I_0 - D + \frac{S}{r_1}$$

$$t_1 = \frac{I_0 - D}{S} + \frac{1}{r_1} \tag{6.2.15}$$

It can be proved that t_1 is positive because this situation exists only under the condition (6.2.10) assumed in this case
Proof:
From (6.2.10) follows:

$$I_0 - D + \frac{S}{r_1} > 0 \qquad /:S$$

$$\frac{I_0 - D}{S} + \frac{1}{r_1} > 0$$

$$t_1 > 0$$

Until the moment t_1 the optimal inventory is given with the expression (6.2.14). From that moment, the problem can be observed like a new one, that begins in the moment t_1 and has the new initial inventory given by

$$I^*(t_1) = D - \frac{S}{r_1} \tag{6.2.16}$$

From that moment further, since the initial inventory satisfies the condition (6.2.10), the solution will be interior. It is important to notice that, because the initial moment for the second part of the problem is no longer zero but t_1, the time translation $t-t_1$ must be introduced. Finally, it gives the expression for optimal inventory in this part of the problem as follows:

$$I^{*}{}' = (I^{*}{}'(0) - D)e^{r_1(t-t_1)} + D$$
$$= (I^{*}(t_1) - D)e^{r_1(t-t_1)} + D \tag{6.2.17}$$

When the (6.2.16) is introduced it gives

$$I^{*}{}' = (I^{*}{}'(0) - D)e^{r_1(t-t_1)} + D \tag{6.2.18}$$

The optimal path for production can be deduced in a similar way:

$$P^{*}{}' = r_1(I'(0) - D)e^{r_1(t-t_1)} + S$$

$$= r_1(I^{*}(t_1) - D)e^{r_1(t-t_1)} + S$$

$$= r_1(-\frac{S}{r_1})e^{r_1(t-t_1)} + S$$

$$P^{*}{}' = S[1 - e^{r_1(t-t_1)}] \tag{6.2.19}$$

Finally, both optimal paths for inventory and production are given by following equations:

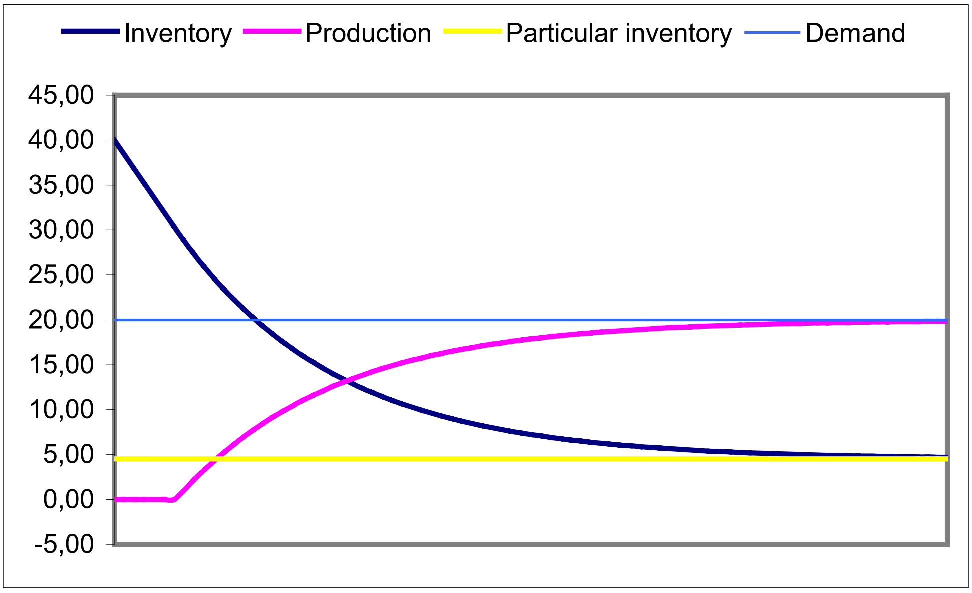
$$I^{*} = \begin{cases} I_o - St & 0 \le t \le \frac{I_0 - D}{S} + \frac{1}{r_1} \\ -\frac{S}{r_1}e^{r_1(t-t_1)} + D & t > \frac{I_0 - D}{S} + \frac{1}{r_1} \end{cases}$$

(6.2.20)

$$P^{*} = \begin{cases} 0 & 0 \le t \le \frac{I_0 - D}{S} + \frac{1}{r_1} \\ S[1 - e^{r_1(t-t_1)}] & t > \frac{I_0 - D}{S} + \frac{1}{r_1} \end{cases}$$

12) Example

$I_0=40>30{,}115528$ and the other parameters of the model remain the same



This example illustrates the situation when the initial inventory is so high that it exceeds the critical value $D-\frac{S}{r_1}$ which, as it was shown in case 2.3, causes boundary solution for optimal production. So, optimal decision rule (6.2.9), as applied in the beginning of planning horizon, gives production equal to zero (boundary solution) and the decrease of inventory w following the demand rate. It proceeds so until the moment t_1, given by (6.2.15), in which inventory level reaches value of $D-\frac{S}{r_1}=30{,}115528$. From that moment further, the optimal solution for production is again interior, and it follows the equation (6.2.20).

The purpose of outlined analysis is to show the dependence of the optimal solution of the model on the initial inventory level as to whether it has the boundary solution or not, which enables to acquire the criteria for management decision.

6.3 Generalization of the basic model by introducing nonnegativity constraints on state and control variable and nonconstant desired levels of inventory and production

In this section, I will introduce and set the optimal linear decision rule for production and the two point boundary value problem (TPBVP) for the most general model, where demand and the desired levels of production and inventory are no longer constants, but both are constrained to be nonnegative. I will do that because in a real world system, the desired inventory and production paths $(\hat{I},\hat{P})$ are very rarely constant and it is necessary introduce them as a functions of time $(\hat{I}(t),\hat{P}(t),t\in[0,T])$ in order to get the model closer to reality. Nonnegativity constraint on production assumes that the firm is either producing (P>0) or not (P=0) and that what is needed today cannot be produced tomorrow. The nonnegativity constraint on inventory says that there is no backlodging.

The symbols for desired production and inventory $\hat{P}$ and $\hat{I}$, are same as before, but now they are functions of time, $\hat{P}(t)$ and $\hat{I}(t)$. These, along with the demand rate, are assumed positive and continuously differentiable. The optimal control problem of the model is then given by:

$$\max J = -\int_0^T e^{-\rho t}\{a[I(t)-\hat{I}(t)]^2 + hI(t) + b[P(t)-\hat{P}(t)]^2 + pP(t)\}dt$$

$$\dot{I} = P(t) - S(t)$$
$$I(t) \geq 0$$
$$P(t) \geq 0$$

the current value Hamiltonian is defined in the usual way

$$H = -a[I(t)-I(t)]^2 - hI(t) - b[P(t)-\hat{P}(t)]^2 - pP(t) + \lambda(t)[P(t)-S(t)]$$

Because of the constraint on the state variable I(t)>0 the Lagrangeian will be introduced

$$\alpha = H + \mu(t)I(t)$$

where $\mu(t)$ is a Kuhn Tucker multiplier

6.3.1 Necessary and sufficient conditions for optimality

Let $(I^*(t), P^*(t))$ be solution paths to this problem. Using the Pontryagin's maximum principle, the following theorem then provides necessary and sufficient condition for optimality:

Theorem 2

In order that $(I^*(t), P^*(t))$ be optimal solutions paths for given optimal control problem, it is necessary that there exists a piecewise continuous function $\lambda(t)$, where for all $0 \le t \le T$ there is $\lambda(t) \ge 0$ such that for every $0 \le t \le T$ the following conditions are fulfilled:

1)

$$\max_{P(t)\ge 0}\{-b[P(t)-\hat{P}(t)]^2 + \lambda(t)P(t) - pP(t)\} = -b[P^*(t)-\hat{P}(t)]^2 + \lambda(t)P^*(t) - pP^*(t) \tag{6.3.1}$$

2) $$\dot{\lambda} = -\frac{\partial \alpha}{\partial I(t)} + \rho\lambda(t)$$

$$= 2a[I(t) - \hat{I}(t)] + h - \mu(t) + \rho\lambda(t) \tag{6.3.2}$$

3) $\lambda(T) \ge 0 \qquad \lambda(T)I^*(T) = 0$ (6.3.3)

4) $\mu(t) \ge 0 \qquad \mu(t)I^*(t) = 0$ (6.3.4)

The function $\lambda(t)$ is the adjoint or costate variable that measures the shadow price of inventory. The function $\mu(t)$ is a Kuhn-Tucker multiplier associated to the non-negativity constraint of the inventory.

6.3.2 Explanation of conditions

The first condition, given by (6.3.1), is the condition of maximizing Hamiltonian (or Lagrangeian) with respect to $P(t) \ge 0$. The first derivative of Lagrangeian cannot be used because, since there is a nonnegativity constraint on control P(t), a boundary solution can occur and it would not be included in that condition.

The second condition (6.3.2) presents an equation of motion for the adjoint variable. Since the optimal control problem has a discount factor, the current value Lagrangeian is used. This is why there is an aditive term $\rho\lambda$ in this equation.

The third condition (6.3.3) is transfersality condition for the state variable I(t) for truncated vertical terminal line (terminal time T is fixed, terminal state can vary, but it has maximum or minimum permissible level, in this case I(T)$\geq$0). In this situation, only two types of outcome are possible in the optimal solution:

$$I^*(T) > 0 \text{ or } I^*(T) = 0$$

In the former outcome, the terminal restriction is automatically satisfied. Thus the transversality condition for the problem with regular vertical terminal line would apply:

$$\lambda(T) = 0 \text{ for } I^*(T) > 0$$

In the latter outcome, it follows $I^*(T) = 0$.

Combining these gives the transversality condition for truncated vertical terminal line given with condition (6.3.3) and the nonnegativity restriction of the state variable (I(t) ≥ 0 for every t, then it is I(T) ≥ 0) in the statement of the problem.

It represents the familiar complementary-slackness condition from the Kuhn-Tucker conditions. In numerical examples it is always possible to try first the ordinary vertical terminal line condition $(\lambda(T) = 0)$, and check whether the result $I^*(T)$ satisfies the terminal restriction $I^*(T) \geq 0$. If it does, the problem is solved. If not, then $I^*(T)$ must be set to zero, and the problem should be treated as one with a given terminal point, which has no transfersality condition.

The fourth condition given by (6.3.4) together with the given restriction of nonnegativity of the state variable ($I(t) \geq 0$) is again a complementary slackness condition $\mu \frac{\partial \alpha}{\partial \mu} = 0$ i.e., $\mu I^* = 0$ which ensures that term μI in Lagrangeian will disappear in the optimal solution, so that value of $\alpha = H + \mu I$ will be identical with value of H after maximization.

6.3.3 Setting of the optimal linear decision rule for production and the two point boundary value problem (TPBVP) for the model

From the condition (1), the following can be concluded:

If the optimal production rate is positive over the planning horizon, then we are dealing with interior solutions and the first-order condition $\frac{\partial \alpha}{\partial P} = 0$ can be used. It gives:

$$-2b(P^*(t) - \hat{P}(t)) - p + \lambda(t) = 0 \qquad (6.3.5)$$

From (6.3.5) $\lambda(t)$ can be expressed as

$$\lambda(t) = 2b(P^*(t) - \hat{P}(t)) + p \qquad (6.3.6)$$

and $P^*(t)$ can be written as:

$$P^*(t) = \frac{1}{2b}(\lambda(t) - p) + \hat{P}(t) \qquad (6.3.7)$$

Since in this case $P^*(t)$ is assumed to be positive, from (6.3.7) follows

$$\frac{1}{2b}(\lambda(t) - p) + \hat{P}(t) \geq 0$$

or

$$\lambda(t) \geq p - 2b\hat{P}(t)$$

Otherwise, if the optimal production rate is negative, because of restriction on P(t), it must be set to zero. So, following these conclusions, the optimal liner decision rule can be expressed like this:

$$P^*(t) = \begin{cases} 0 & \lambda(t) < p - 2b\hat{P}(t) \\ \frac{1}{2b}(\lambda(t) - p) + \hat{P}(t) & \lambda(t) \geq p - 2b\hat{P}(t) \end{cases} \qquad (6.3.8)$$

Now the derivative of $\lambda(t)$ can be found from (6.3.6)

$$\dot{\lambda}(t) = 2b\overset{\bullet}{P}{}^{*}(t) - 2b\dot{\hat{P}}(t) \qquad (6.3.9)$$

$\lambda(t)$ from (6.3.6) and $\dot{\lambda}(t)$ from (6.3.9) can be substituted in equation of motion for costate variable given in the condition (6.3.2). Because now the optimal solutions are used, zero can be substituted for $\mu(t)$

$$2b\dot{P}^{*}(t) - 2b\dot{\hat{P}}(t) = 2aI^{*}(t) - 2a\hat{I}(t) + h + \rho 2bP^{*}(t) - \rho 2b\hat{P}(t) + \rho p$$

and the following equation is deduced

$$\dot{P}^{*}(t) = \rho P^{*}(t) + \frac{a}{b}I^{*}(t) + \dot{\hat{P}}(t) - \rho\hat{P}(t) - \frac{a}{b}\hat{I}(t) + \frac{\rho p + h}{2b} \qquad (6.3.10)$$

From the equation of motion for the state variable (inventory) given in the statement of the problem, second equation for the system of two differential equations is given as

$$\dot{I}^{*}(t) = P^{*}(t) - S(t) \qquad (6.3.11)$$

The system of two differential equations consisting of (6.3.10) and (6.3.11) can be expressed in the matrix form:

$$\begin{bmatrix} \dot{P}^{*}(t) \\ \dot{I}^{*}(t) \end{bmatrix} = \begin{bmatrix} \rho & \frac{a}{b} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P^{*}(t) \\ I^{*}(t) \end{bmatrix} + \begin{bmatrix} \dot{\hat{P}}(t) - \rho\hat{P}(t) - \frac{a}{b}\hat{I}(t) + \frac{\rho p + h}{2b} \\ -S(t) \end{bmatrix} \qquad (6.3.12)$$

Since for $I^{*}(T)>0$, $\lambda(T) = 0$ is valid, from lower part of linear decision rule (6.3.8), follows:

$$P^{*}(T) = \hat{P}(T) - \frac{p}{2b} \qquad (6.3.13)$$

It is the terminal condition. The equation (6.3.13), considered along with initial condition $I(0) = I_0$ and the system of differential equation (6.3.12), create two point boundary value problem (TPBVP) that should have been set up.

7. ECONOMIC INTERPRETATION OF THE MODEL

7.1 Introduction

In the introductory chapter, I stated that the purpose of developing and analysis of mathematical models is to provide a better insight to the problem and to help managers in making better business decisions. That is why the understanding of the economic meaning behind the symbols and formulas used in them is very important and helpful. Since the maximum principle was used for the analysis and optimization of the given model, I will explain variables and conditions it imposes.

This analysis will also show the sensitivity of the model to some values, which provides quantitative analysis of change of the optimal solutions and of the value of optimal criterial functional, depending on change of variable values. Using this information, manager can better estimate the cost of a particular change as well as whether it is profitable or not. By doing so, he can also learn information where the model is more sensitive to changes, because it is there that he must be more careful when making decisions or even introduce some kind of safety mechanism.

For this analysis, I will use some formulas and reasoning from the paragraph 5.5 where the rationale of maximum principle was exposed.

As mentioned before, generally, there are three types of variables: control, state and costate. I will interpret them first.

7.2 Economic interpretation of P(t), I(t) and λ(t)

In the given model, the state variable is inventory stock level I(t). There is also a control variable, rate of production P(t), representing business decision, that manager has to make at any moment of time. The firm starts at zero time with a given inventory stock I_0 but the terminal inventory stock is not determined. At any moment of time, the costs C(t,I,P) given by (6.2') that the firm wants to minimize, depend on the amount of inventory it holds, as well as on the level of production P it currently selects.

Further, the selection of the level of production influences the rate at which inventory stock I(t) changes over time. It means that $\dot{I}$ is affected by P, which is included in the equation of motion for the state variable (6.1). So, the economic meaning of the control variable (or it can be called decision variable) P(t) and the state variable I(t) is self-explanatory, but the meaning of costate is not.

Since the costate variable λ(t) was introduced in the Hamiltonian in the nature of Lagrange multiplier in (5.5.3), it should have the connotation of a shadow price. To see this, I will use expression (5.5.7) and introduce the optimal paths for all variables in it. I will adopt it in the given model, which, initially, was a minimization model, and because of that the cost function in Hamiltonian has a minus sign. I will also take a discount rate into account. After such a modification, the optimal functional looks like this

$$\max J'^{*} = \int_0^T [H(t, I^*, P^*, \lambda^*) + I^* \dot{\lambda}^*] dt - \lambda^*(T) I^*(T) + \lambda^*(0) I_0 \qquad (7.1)$$

Partial differentiation of J'^{*} with respect to the known initial inventory I_0 gives:

$$\frac{\partial J'^{*}}{\partial I_0} = \lambda^*(0) \qquad (7.2)$$

From the equation (7.2) it is obvious that $\lambda^*(0)$, which is the value of optimal costate variable in the initial moment, is the measure of sensitivity of the total optimal production – inventory cost to the given initial inventory stock. Or, it can also be interpreted like this: if there had been one more (infinitesimal) unit of inventory initially, the optimal total costs would have been smaller for $\lambda^*(0)$ (Because Hamiltonian in given model has the sign opposite to the costs sign). So, the $\lambda^*(0)$ can be viewed as an imputed value or a shadow price of the initial inventory.
Similarly, (7.1) can be partially derived with respect to the optimal terminal inventory $I^*(T)$, giving:

$$\frac{\partial J'^{*}}{\partial I^*(T)} = -\lambda^*(T) \qquad (7.3)$$

In this equation, the value of the optimal costate function in the terminal time $\lambda^*(T)$ has the sign opposite to that of the partial derivative. (In the given model, the same sign as costs).

Since the partial derivative measures the sensitivity of the optimal value of total negative costs (J'^*) on the optimal terminal inventory stock $I^*(T)$, the conclusion is the following: If a firm wants to have one more unit of inventory stock at the end of the planning horizon, it will have the optimal total costs increased by the amount of $\lambda^*(T)$. So, in this case $\lambda^*(T)$ is also the shadow price of a unit of inventory at the terminal time. When these observations are gathered, the conclusion is that $\lambda^*(t)$ generally measures the sensitivity of optimal total cost on the inventory stock or it can be said that $\lambda^*(t)$ is the shadow price of inventory at that particular point of time.

7.3 Economic interpretation of Hamiltonian and the condition of maximization

The Hamiltonian in a given model is

$$H = -e^{\rho t}[C(t, I(t), P(t))] + \lambda(t) f(t, I, P) \tag{7.4}$$

The first term on the right side of (7.4) is, as it was shown before, the discounted negative value of the production-inventory cost function at time t. It is based on the current production and inventory policy decision made at that time. It actually represents the negative value of the “current costs corresponding to policy P”.

Second term on the right hand side of (7.4) is the equation of motion of state variable $(f(t, I, P) = \dot{I} = P - S)$, which measures the rate of change of inventory stock I (depending on used policy P), but here it is multiplied by the shadow price $\lambda(t)$. Because of that it measures a monetary value. So, the second component of the Hamiltonian represents the “rate of change of the value of inventory level corresponding to chosen policy of production”.

It can also be interpreted as the future costs effect of production policy, because P influences future stock of I, which then again influences the future costs. The influences of these two terms are competing in nature. If a chosen production decision, for example, is favorable to the current cost, it will be less favorable for the future costs. So, the conclusion is that Hamiltonian represents the expected overall costs of the various production decisions with both current and future costs included.

The first condition of the maximum principle is maximization of Hamiltonian (in presented model it means minimization of costs) with respect to control variable (in our case production P). In other words, it requires that a firm, at each point of time, chooses the proper decision for production following the goal of achieving the lowest possible overall expected costs a.

This requires the proper balancing of expected savings in the current costs against expected losses in future costs. It will be more obvious from the "weak" version of the condition for maximization:

$$\frac{\partial H}{\partial P} = 0$$

$$-e^{-\rho t}\,\frac{\partial C(t,I,P)}{\partial P} + \lambda(t)\frac{\partial f}{\partial P} = 0$$

Rewritten as follows

$$e^{-\rho t}\,\frac{\partial C(t,I,P)}{\partial P} = \lambda(t)\frac{\partial f}{\partial P} \tag{7.5}$$

it shows that the optimal choice of production P must balance the discounted marginal change in the current cost caused by that choice (left side of (7.5)) with the marginal change of the future costs which will be caused by P indirectly trough the marginal contribution of P to the change of inventory stock captured in the equation of motion (the right side of equation (7.5))

7.4 Economic interpretation of equations of motion

The other two conditions in maximum principle are called equations of motion. The one that concerns the state variable I(t) is included in the problem statement and it describes how the chosen policy for production P(t) influences the rate of change (or the motion) of the inventory level I(t).

$$\frac{\partial H}{\partial \lambda} = f(t, I, P) = \dot{I} \qquad (7.6)$$

The other equation of motion is for the costate variable

$$\dot{\lambda} = -\frac{\partial H}{\partial I} = e^{-\rho t}\frac{\partial C(t, I, P)}{\partial I} - \lambda(t)\frac{\partial f}{\partial I} \qquad (7.7)$$

In a given model, the equation $\dot{I}(t) = f(t, I(t), P(t)) = P(t) - S(t)$ is valid and therefore the second expression from the right hand side of (7.7) disappears.

The value of $\dot{\lambda}$, from the left side of equation (7.7), represents the rate of change of the shadow price for inventory over time. So the given equation of motion shows that this rate must be of equal magnitude as discounted marginal contribution of inventory to the current costs. In other words, the maximum principle in a given model requires that the shadow price of inventory increases at the rate at which inventory is contributing to the inventory-production costs, but discounted.

8. CONCLUSION

In this master's thesis, the simple HMMS type of model was upgraded by adding a new type of costs and the continuous discounting. Then, the optimization and analysis of the new model, using the optimal control theory methods, was performed.

The HMMS type of models minimizes costs of deviation of the production and inventory levels from their goal values (quadratic costs), which in some situations may cause a huge deviation from reality because they cannot handle situations where production or inventory has a high level, but pretty close to the desired. In such situations, HMMS model would have very small or zero costs for it, which is far from reality.

This is avoided by introducing a linear costs for regular production and keeping inventory, and distinguishing these costs in respect to costs of deviations from the desired levels, which, in this model, are named extra costs.

The analysis of the model and examples has shown that these two kinds of costs influence behavior of the optimal paths in different ways, which is also a good reason for introducing them.

By implementing continuous discounting to the model, extension of the planning horizon to a long or infinite time interval is enabled, because it considers that discounted returns from the far future become negligible.

Since the area of production and inventory control is an integral part of a new field called supply chain management (SCM), I first introduced this problem and presented respective explanations.

In the second chapter, I presented a brief historical overview of production and inventory models, including the optimal control models.

Then I overviewed several important contemporary methods of managing inventory and production logistics.

Since the mathematical foundation of this paper is the dynamic optimization, in the fifth chapter, I exhibited it in the light of three most widely used methods and I compared them.

In Chapter six, I developed, optimized and analyzed, theoretically and practically, through the examples, above explained model. In the end of this chapter, I set up the linear decision rule and two-point boundary value problem (TPBVP) for the most general version of the model presented.

Finally, in the seventh chapter, I proposed an economic interpretation of variables and equations of the model, providing in the same time some sensitivity analysis. The purpose was to provide a better insight to the model and to help a manager who would like to use the model as guidance in making decision**,** to understand better the meaning of variables and equations, relationships between them and sensitivity of optimal solutions in respect to the main variables.

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