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AN EMPIRICAL COMPARISON OF STRUCTURAL CREDIT RISK MODELS

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ANDREJ SEDEJ

AUTHORSHIP STATEMENT

The undersigned Andrej Sedej a student at the University of Ljubljana, Faculty of Economics, (hereafter: FELU), author of this written final work of studies with the title An Empirical Comparison of Structural Credit Risk Models, prepared under supervision of prof. dr. Igor Lončarski

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INTRODUCTION

"A firm defaults when it fails to service its debt obligations," is how Vassalou and Xing (2004) define a default of a company. Estimation of a company's probability of default (also PD) is one of the most important tasks relevant to banks, rating agencies, regulators, and other financial institutions. That said, default forecasting also enables investors to better determine spreads over the risk-free interest rates, rate a firm, decide whether a credit should be given or not, and helps them to better manage their credit risk. Beinstein and Scott (2006) define the latter as the risk of a loss that an investor could suffer in case the issuer goes into default.

Since plenty of innovative debt and credit derivative products emerged in the past three decades, researchers have recently shown an increasing interest in the PD models (Vassalou & Xing, 2004; Bharath & Shumway, 2008). Indeed, an inaccurate estimation of the PD can lead towards a large mispricing of financial instruments such as credit default swaps (also CDS) or corporate bonds. Apart from it, this can contribute to fallacious credit ratings which was one of the many reasons for the most recent global financial crisis in 2007–2009 (Gurny, Lozza & Giacometti, 2013).

The occurrence of the financial crisis, when many financial institutions failed, resulted in regulators more than ever instructing banks and other financial institutions to constantly maintain at least a predetermined value of equity. By doing so, the regulators attempt to avoid such events in the future (Imerman, 2011). Due to Basel regulations, financial institutions have to ensure that their credit risk models can be empirically supported. What is more, the international regulatory frameworks for banking institutions, Basel II and Basel III, impose the probability of default models for the calculations of regulatory capital. Indeed, the probability of default represents a key parameter in Basel standards (Gurny, Lozza & Giacometti, 2013; Suo & Wang, 2006).

There are several ways of PD estimation; however they can generally be distributed into three major groups. Credit-scoring models are considered to be the most traditional group among the probability of default measurement techniques, which has already been used by Altman (1968). A well-known credit-score modelling approach is a simple linear regression model, however, its assumptions are disputed. In particular, it does not take into account the lower and upper boundaries of the probability measure, meaning 0 and 1, respectively. Non-linear models, such as probit and logit regression models, are thus regarded as reasonable alternatives. Each of the aforementioned models includes several explanatory variables that indicate whether an individual or a company is going to default in a predetermined time or not. In general, the credit-scoring models are most widely used and considered supreme especially in retail credit markets, where the data is not updated on a frequent basis and is generally limited to the balance sheet data rather than market information (Allen, DeLong & Saunders, 2003).

An alternative approach of the PD estimation is observed in the family of reduced-form models. Among this kind of models, the default time is defined as the first jump time of a Poisson process with a deterministic or stochastic intensity. Hence, the default is not a result of market information but rather a consequence of an exogenous effect, which is independent of the market movements. Such type of modelling may especially be useful when determining credit spreads as it is easy to calibrate to the CDS data (Brigo & Tarenghi, 2005).

The other set of alternative credit risk models is the structural credit risk model family. One of the main aims of the structural credit risk models is to capture the nonlinear relationship of the inputs with respect to the implied PD estimates. Within the scope of this thesis I will only concentrate on the latter type of models. The first structural credit risk model emerged in 1974 when Robert Merton developed a brand new concept for the pricing of risky corporate bonds. The inspiration for this model comes directly from the European option pricing model, which was proposed a year earlier by Black and Scholes (1973).

In his paper Merton (1974) views the equity of a firm as a European call option on its own assets. The basic logic of the model is reasoned since, according to his assumptions, shareholders are entitled to the difference between a variable amount of assets and the fixed obligations of the firm. Clearly, this is valid only if the difference is non-negative as share price cannot turn negative. This micro-economic interpretation of the default represents the main advantage of the Merton and the structural credit risk models in general (Afik, Arad & Galil, 2012; Katz & Shokhirev, 2010).

Consequently, the Black-Scholes option pricing model can also be applied to estimation of the default probability. In particular, in Merton's framework, the probability of default is determined as the probability that the call option will not be exercised at the maturity of obligations. Due to simplicity of the model, the PD formula takes an analytical form (Imerman, 2011). However, very soon many theoretical studies followed as researchers aimed to relax a few of the unnecessary and even unrealistic assumptions presented in the Merton's paper. As an evident corollary, the upcoming structural credit risk models became more complex both in their derivations and implementations.

The most criticized assumption in the base model is the assumption that all obligations mature at the same time and that the default can only take place at the maturity of these obligations. Thus, the original structural model does not consider the dynamics of assets in their lifetime, it only takes into account their value at the maturity, which may be an oversimplification. That is what Black and Cox (1976) tackle in their paper where they introduce a safety barrier which represents the amount of firm's obligations. In case that assets fall below this exogenously specified barrier any time before the maturity of the debt, the default event arises. Hence, they do not see the firm's equity value as a vanilla call option anymore but rather as a down-and-out European barrier call option on assets. Henceforth, they create a credit risk model where the default time itself becomes uncertain as well.

Moreover, Longstaff and Schwartz (1995) make a step further. They use a flat default barrier which is a special case of the Black and Cox (1976) idea. However, they relax the assumption of constant interest rates, consequently producing a two-factor model. In this model, the probability of default does not take a closed-form solution, therefore, numerical methods or approximations have to be applied (Imerman, 2011).

The recent research rise of the structural credit risk modelling led to another important extension of the Longstaff-Schwartz model by Collin-Dufresne and Goldstein (2001). As opposed to the other models, where the barrier is deterministic or even constant, their paper introduces a mean-reversion for the safety barrier. Hence, the leverage of the company oscillates around the target ratio. Another unreal assumption in the initial Merton and other aforementioned models is that the firm's value of assets follows the Geometrical Brownian motion, which does not allow for sudden jumps of assets and quick spikes in volatility. Therefore, none of the mentioned models perfectly captures the behavior on the market where such events are not uncommon. As a result, Huang and Huang (2002) introduce a double exponential jump diffusion barrier model which takes the Geometrical Brownian motion as the base for the asset's value and combines it with the Poisson-like distributed occurrence of jumps.

On the other hand, the KMV model can be regarded as a benchmark for practical implementations of the initial Merton model as it is widely used in the financial sector. The model distinguishes between a short-term and long-term debt of a given company and takes a linear combination of the two in order to define the default threshold at the maturity (Imerman, 2011). An important limitation that has been observed by other researchers is that Moody's KMV utilizes its large database, which may not be accessible to other practitioners. Moody's database accounts for more than 2000 defaults which enables them to obtain the empirical distribution of distance to default, which is one of the important elements for the PD computations (Vassalou & Xing, 2004). Since this modelling data is an internal information, researchers usually use a cumulative normal distribution as an approximation for purposes of converting the distance to default to the probability of default. As a result, Bharath and Shumway (2008) claim they could not perfectly replicate the original KMV's methodology.

Alternatively, there exists a market-based approach which relies on the assumption of market efficiency. By using this probability of default estimation technique, one can imply the PD from the quotes of market instruments such as corporate and sovereign bonds or the CDS. Many claim that the latter financial instrument provides a better credit risk information, however, according to Chan-Lau (2006) these are not always available. On the other hand, the implication of PD and credit spreads out of the bond yields necessarily involves an assumption related to the risk-free rate benchmark, which may importantly affect the accuracy of the estimation (Hull, Predescu & White, 2004b). That said, the market efficiency and lack of data might indeed compromise the results of my study. In order to limit these effects, I do not analyze structural credit risk models on the Slovenian market where the amount of quoted and liquid companies may be low. Hence, in order to be closer to fulfilling the efficient market hypothesis, the focus of this thesis will be on the US companies, which formed S&P500 index on a specific date in the past.

According to the past empirical studies it has been observed that the structural approach does not deliver well, especially when it comes to the pricing of risky obligations or the generation of credit spreads. On the other hand, they perform better when it comes to the prediction of corporate default and the calculations of default probability (Jones, Mason & Rosenfeld, 1984).

A large portion of literature on structural credit risk models is focused on the variations of basic Merton and KMV models. A few of the first researchers that investigate the Merton model's contribution are Moody's practitioners (Sobehart & Stein, 2001; Stein, 2005). They conclude that the model can easily be improved, which is also supported by Gurny, Lozza and Giacometti (2013) study. The latter researchers suggest that the PDs are in general underestimated in the Merton model, however, Duffie, Saita and Wang (2007) argue that Merton's PD are significant and therefore useful. Bharath and Shumway (2008) examine whether the PD estimate obtained by the variation of the KMV model, namely the Merton DD, can improve the forecasting power of the traditional linear regression and hazard models. Following the comparison of the Merton DD with their Naïve model, which still keeps Merton's functional form, they conclude that Merton's PD estimate is a useful but not sufficient variable for forecasting the time to default. Nevertheless, they observed that the usefulness comes from the Merton model's functional form rather than the calibrations used in the original KMV model.

Apart from that, Jones, Mason and Rosenfeld (1984) as well as Huang and Huang (2002) test whether bond yield spreads can be explained with the existing structural credit risk models. In particular, Huang and Huang (2002) study a range of them and they conclude that the observed models generally produce too low yield spreads for less risky bonds. On the other hand, they perform better for riskier products.

Furthermore, Huang and Zhou (2008) compare Merton, Black-Cox, Longstaff-Schwartz, Collin-Dufresne Goldstein, and Huang and Huang models. In particular, they focus only on the implications of structural models to credit default swap pricing, however, they do not analyze the time to default forecasting. In their study they observe that the complexity of the model improves a forecasting performance as the Collin-Dufresne Goldstein and Huang and Huang models significantly outperform the other three models. Nonetheless, each of the studied models still has difficulties when predicting credit spreads.

In a similar manner, Suo and Wang (2006) also empirically study a range of structural credit risk models and conclude that for the less risky firms the Merton model predicts annual PDs too close to 0, and hence underestimates the risk of default. On the other hand, based on Suo and Wang (2006), PDs produced by Longstaff-Schwartz model are in average close to Moody's and the S&P default probabilities. However, the Collin-Dufresne and Goldstein model overestimates the default probability with respect to the two mentioned benchmarks. Nonetheless, authors emphasize that even though the more complex models generally outperform the Merton model, they introduce additional sensitivity related to the interest rate and safety barrier parameters.

Overall, one can claim that there are several research papers that are studying the variations of Merton models only or explaining bond yields through a spectrum of structural credit risk models. Generally, most of them scrutinize the simplest Merton model and claim that the complexity positively influences the model's forecasting performance. However, according to my knowledge there does not exist a study, which would compare the more complex structural models based on their time to default forecasting performance. Bharath and Shumway (2008) perform such research but only focus on the Merton model variations. Part of the reason, why this kind of research may not have been done yet, is that some models are still relatively new. But even more importantly, they are not as widely used as, for example, the traditional regression models are.

Therefore, most researchers came to the conclusion that the initial Merton model's assumptions are not realistic enough for default prediction on its own. But how its extensions perform when it comes to the forecasting of time to default? Within the scope of my master's thesis, the main task will be to implement Merton, Black-Cox, Longstaff-Schwartz, and Collin-Dufresne Goldstein models and apply the input data in order to produce their PD estimates. In the following, I will be able to compare their estimates of the PD as well as observe which one performs better and by how much the performance of a structural credit risk model improves when increasing the complexity, if at all.

My second aim is to examine through regressions whether a PD given by any of the models is a sufficient predictor for default prediction purposes. In other words, if any additional variable would be a significant predictor of time to default, it would imply that it adds to the forecasting power of the model. Consequently, a given structural credit risk model's PD would not be sufficient. Furthermore, I want to examine whether any of the structural credit models' PD estimators is an important predictor of default. This would be the case if a PD estimate of a given structural credit risk model would be the only significant predictor of the model and remain significant even after adding several other regressors into account. Finally, I will study sensitivities with respect to the input parameters and aim to assess whether any model is more sensitive to the inputs than others.

Therefore, the key questions within the scope of the master's thesis are the following:

1. How sensitive are structural credit risk models with respect to their inputs and how the sensitivities differ between the models?
2. Is any of the structural credit risk models' algorithms a sufficient or at least an important predictor of default?
3. Does the complexity of the structural credit risk model add to the accuracy of the time to default forecasting and, if so, how much?

Within the introduction of the thesis, I highlight the importance of credit risk modelling and present several ways of PD modelling. Additionally, I discuss the main ideas of the most widely known structural credit risk modelling approaches and I summarize other researchers' discoveries. Finally, I point out which are the main objectives of this thesis.

In section 1 I focus on assumptions, theoretical frameworks of the models and parameter calibrations. In addition, I provide rationales, which are important for understanding and assuring the soundness of the studied structural credit risk models. Furthermore, I discuss the Cox proportional hazard regression model that accounts for censorship. I use it in order to assess the structural credit risk models' significance and importance. In addition, I present credit default swaps which are the financial products that carry market information of the probability of default. In section 2, I present the input data along with its corresponding sources, the calibrated variables and compare the summary statistics of the implied PDs. Moreover, I briefly discuss what additional or alternative input data might be needed in case of enhanced studies and how it would be used. Additionally, I debate modelling choices used in order to implement the models. In section 3, I perform a sensitivity analysis on structural credit risk estimates, investigate, which model is more influenced by the shifts of parameters, and discuss the rationale for it. Besides, I display and interpret empirical results of tests and regressions performed. In conclusion I attempt to answer the key questions, which are specified in the introduction of the thesis, and summarize the key results.

1 THEORETICAL FRAMEWORK

Fundamentally, the structural credit risk models rely on a basic accounting equation. In particular, the market value of an equity E equals the difference between the market value of assets V and the debt F. Nevertheless, in order to avoid the concept of negative equity, this type of models generally assume that once the difference between the market values of assets and debt drops below 0, the default occurs (Mišankova, Kočišova & Klištik, 2014). The market value of equity is public information and, therefore, available to every investor. However, the important limitation of every single structural credit risk model is the unavailability of market prices of assets.

Structural credit risk models are unusual in the financial industry as the majority of forecasting models are based on estimation techniques such as the maximum likelihood or the generalized method of moments. On the other hand, the practical implementations of structural credit risk models rely on the calibrations of implied parameters (Bharath & Shumway, 2008). Generally, there is a trade-off between the simplicity of the model and the soundness of assumptions in the structural credit risk model framework. In this section, I first present the models that rely on simplicity and I progress towards the more complex models later on. Apart from that, in subsection 1.5 I present the Cox proportional hazard regression model that I utilize as a method to compare the structural credit risk models. Finally, in section 1.6 I briefly describe the model, which transforms the CDS spreads into the market implied PDs. The topic is relevant as I use CDS implied PDs in section 3.3 in order to challenge the importance of structural credit risk models.

1.1 Merton Model

The initial and the most basic theoretical structural credit risk model is the Merton model. The Merton model treats the equity of a firm as a European call option on the value of assets. This logic is valid since, according to this model's assumptions, shareholders are entitled to the difference between the variable value of assets and fixed obligations of the firm. However, this is only reasonable if the difference is non-negative as share price cannot be negative. As a result, it is convenient to make use of the European option pricing idea that was introduced by Black and Scholes in 1973.

In order to enable the use of the Black-Scholes model type, Robert C. Merton had to assume the following (Merton, 1974; Bharath & Shumway, 2008; Sundaresan, 2013):

1) No transaction costs or taxes and a perfect divisibility of assets.
2) Investors can buy or sell indefinite amount of assets.

3) Market participants can borrow and lend at the same risk-free rate r and they are well informed.
4) Short-selling is allowed.
5) Trading of assets is continuous.
6) The instantaneous riskless rate r is constant and known in advance.
7) The asset return volatility σ_V is constant till the credit's maturity.
8) The firm has issued debt, a homogenous liability, which can also be considered as a zero-coupon bond with a notional *F* that matures at a maturity *T*. Therefore, the default of a firm can happen no earlier than at its maturity *T*.
9) The total value of a company's assets under the physical measure can be described with a stochastic process that follows the geometric Brownian motion which is defined by the stochastic differential equation (SDE) below.

$$dV_t = \mu V_t dt + \sigma_V V_t dW_t \tag{1}$$

In equation (1) V_t represents the total asset value in time *t*, μ and σ_V are the instantaneous expected value and standard deviation of asset return rate, respectively, and $\{W_t\}$ stands for the standard Wiener process.

If the above assumptions are not disputed, one can model equity of a firm E as a European call option with the strike being equal to debt of a firm *F*, with underlying of the option matching the total value of assets *V* and the maturity being set to *T*.

$$E(T, V_T) = \max(V_T - F, 0) \tag{2}$$

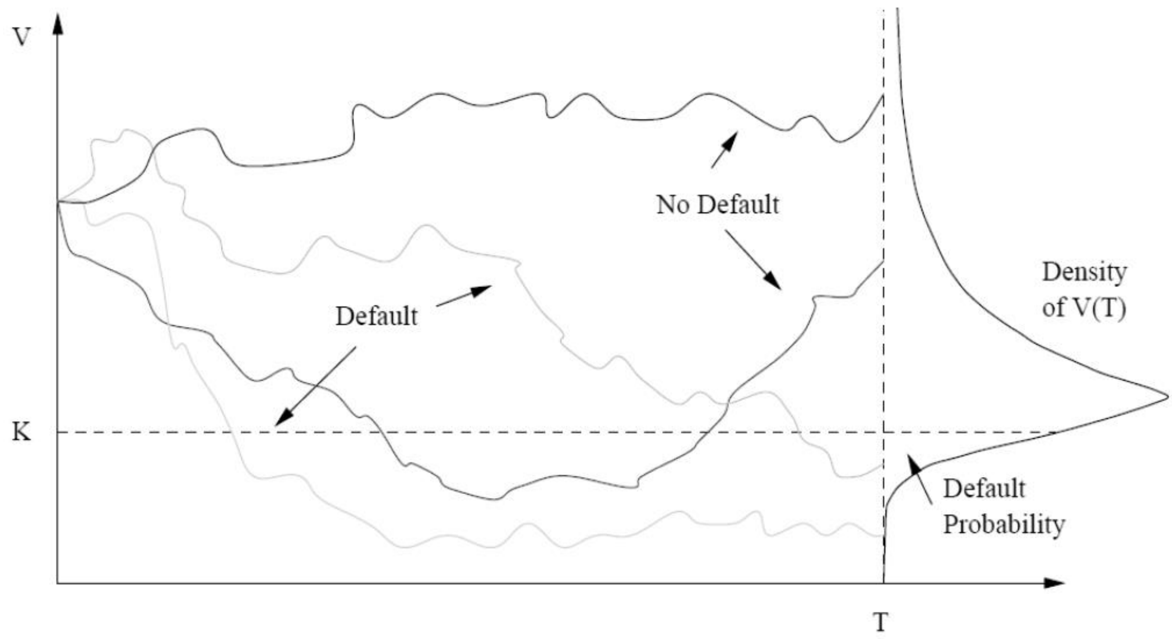
First four assumptions of the model imply the perfect market, whereas the assumptions 6 and 7 substantially simplify the reality like the Black-Scholes model does. Moreover, the assumptions 5, 8 and 9 are crucial to allow the direct application of the Black-Scholes formula when computing equity value at the maturity as a function of the asset price. In particular, due to the assumption 8 one can view the equity of a firm as a European call option rather than any other option type (Merton, 1974).

At the maturity *T*, a firm must pay the promised payment to their own debtholders. If a company does not possess enough assets, it goes into default and partially pays off its debtholders with all its remaining assets V_T. Thus, the payoff of creditors at the maturity can be expressed with the following formula (3) and, consequently, in the framework of this model it can be modelled as a European put option with the strike *F* and the maturity *T* (Merton, 1974).

$$H(T, V_T) = \min(V_T, F) \tag{3}$$

The option pricing idea, which is acquired in the Merton model, is displayed in Figure 1. The firm goes into default for the two trajectories that fall below the obligations level *K* at the maturity *T*. That said, the firm remains solvent in the other two cases even though one of them dropped below the threshold in the forecasting period.

Figure 1: Graphical presentation of asset market value trajectories in Merton model with obligations K and forecasting period T.



Source: Mišankova, Kočišova and Klištik (2014), p. 287.

1.1.1 Probability of Default Formula Derivation

In this subsection, I show the stochastic differential equation derivation of the Merton model which leads to the final PD formula. Let me emphasize that for the sake of exposing the partial derivatives of equity with respect to time *t* and asset value *V*, in SDE equations (6) and (8), I exceptionally use the notation E and V rather than E_t and V_t for the value of equity and assets in time t.

One could replicate a firm's equity at time $t < T$ by holding a portfolio that consists of a W_1 percentage of imaginary security *V*, which perfectly replicates a firm's asset value, and a W_2 share of riskless bonds *B*. Such portfolio could be presented with the following equation (Merton, 1974).

$$\Pi_t = W_1 V_t + B_t \tag{4}$$

Hence, its dynamics is described by (5):

$$d\Pi_t = W_1 dV_t + B_t r dt \qquad (5)$$

where *r* represents a constant instantaneous risk-free rate (Merton, 1974). On the other hand, since in the Merton model the value of a firm $E(t, V_t)$ is a function of time and market value of assets, one can make use of Ito's lemma and thus:

$$dE = E_t dt + E_V dV + \frac{1}{2} E_{VV} (dV)^2 = \left(E_t + \frac{1}{2} E_{VV} \sigma_V^2 V^2 \right) dt + E_V dV \qquad (6)$$

Since the portfolio Π is replicating a firm's equity, by the non-arbitrage assumption, the portfolio price must equal a firm's equity at any point in time. Consequently, both dynamics must be aligned as well (Merton, 1974). This condition results in:

$$W_1 = E_V \qquad (7)$$

Once the above equation is taken into account, one finally ends up with a stochastic differential equation

$$E_t + \frac{1}{2} E_{VV} \sigma_V^2 V_t^2 + E_V V_t r - E r = 0 \qquad (8)$$

subject to its boundary conditions

$$E(t, 0) = 0 \qquad (9)$$

$$E(t, V_t) \leq V_t \qquad (10)$$

Besides, the aforementioned payoff formula (2) represents the third restriction. The condition expressed in the equation (9) implies that the equity level always remains non-negative even if the asset's value equals 0. On the other hand, the equity value can never be larger than the value of a firm's assets, which is what the formula (10) suggests (Merton, 1974).

As in the European option pricing world, the above stochastic differential equation subject to its conditions, presented in the equations (2), (9) and (10), can be analytically solved. It results in the Black-Scholes formula which has been derived in details by Black and Scholes (1973). Thus, the Merton model implies that the value of a firm's equity is calculated as follows (Merton, 1974):

$$E_t = V_t * N(d_1) - e^{-r(T-t)} * F * N(d_2) \tag{11}$$

where

$$d_1 = \frac{\ln\left(\frac{V_t}{F}\right) + \left(r + \frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V\sqrt{(T-t)}} \tag{12}$$

$$d_2 = d_1 - \sigma_V\sqrt{(T-t)} \tag{13}$$

In the Black-Scholes model, the risk-neutral probability that a call option will not be exercised at its maturity, can be expressed as:

$$P(V_T < F) = N(-d_2) \tag{14}$$

However, one can translate the option pricing model to the Merton model. By doing so, the above formula represents exactly the probability of default of a firm at a time t (Bharath & Shumway, 2008).

The Merton model is neither a complex model for the users, as it can be solved analytically, nor very demanding in its derivation or idea. Nevertheless, the assumptions used in the model might be too restrictive which is the main reason for many extensions of the base model in the last few decades (Elizalde, 2006).

1.1.2 The Merton DD Model

The theoretical version of the Merton model produces a few unpleasantries when practitioners attempt to make use of the model and calculate the PD in practice. That is why many practical implementations such as the KMV model or the Merton DD model, which has been studied by Bharath and Shumway (2008), Vassalou and Xing (2004) and other researchers, emerged. Even though the two mentioned practical implementations of the Merton model are rather similar, the main difference is that Moody’s KMV makes use of Moody’s big dataset. That said, it enables Moody’s to estimate the empirical distribution of the distance to default which positively influences the model performance (Vassalou & Xing, 2004).

Apart from that, an important issue faced by practitioners is the computation of the face value of debt F. Implementations of the Merton model usually distinguish a short-term and long-term debt of an observed company and take the linear combination of the two in order to define the default threshold for the assets at the maturity. Short-term liabilities mature by the end of the forecasting period, which usually equals one year. Therefore, they have to be fully included in the model (Imerman, 2011; Bharath & Shumway, 2008). Nevertheless, Vassalou and Xing (2004) argue that even though long-term liabilities may not mature by the end of the forecasting period, they should not be ignored either. The first reason for it is that firms usually have to service their long-term debt and, consequently, these intermediate cash flows count as part of their short-term liabilities. Furthermore, the level of the long-term liabilities has a direct effect on a firm's ability to roll over its short-term debt. What percentage of the long-term should be included in the face value of debt F cannot be easily determined, however, both the KMV and Merton DD models apply a general weight of 50 %. As a result, one approximates the market value of debt as:

$$F = short - term\ liabilities + \frac{1}{2} long - term\ liabilities \tag{15}$$

The other practical problem is the computation of the asset return volatility σ_V. The volatility of assets return is not a market data, however, the equity volatility can be estimated based on historical data or implied from the option market. Furthermore, the market value of assets V_t, which is the main subject of interest, is not directly observable, while the book value of assets is not relevant (Bharath & Shumway, 2008).

Hence, in a similar manner as it has been initially suggested by Jones, Mason and Rosenfeld (1984), the Merton DD model uses two non-linear equations in order to numerically infer the market value of assets and the asset return volatility. Firstly, it takes into account the equity value formula (11) where the equity is a function of both assets and the asset return volatility. The second formula can be derived using Ito's formula as the equity value $E(t, V_t)$ is a function of time and assets in the Merton DD model. Since it is twice differentiable in the Merton model (Bharath & Shumway, 2008):

$$dE = E_t dt + E_V dV_t + \frac{1}{2} E_{VV} \sigma_V^2 V_t^2 dt \tag{16}$$

It follows that

$$\sigma_E = \frac{V}{E} E_V \sigma_V \tag{17}$$

where derivative E_V is nothing else but the call option's delta $\frac{\partial E}{\partial V} = \Delta_C = N(d_1)$. Hence, one ends up with two non-linear equations (11) and (17) in which V_t and σ_V are both unknowns, while other parameters are all observable in the market. The only exception is T which represents the forecasting horizon and is subject to modeler's choice. Henceforth, this set of equations can be solved numerically (Bharath & Shumway, 2008; Chan-Lau, 2006).

For the sake of resolving this problem of two non-linear equations (11) and (17), Vassalou and Xing (2004) and Bharath and Shumway (2008) suggest implementing the following iterative procedure. The initial value of σ_V in the iteration should be set to $\sigma_V = \sigma_E \frac{E}{E+F}$. Thus, the initial step supposes that the volatility of the equity is mostly driven by the asset return volatility σ_V which is close to proportionally lower with respect to σ_E. In the following, the proposed iteration step is to insert the computed σ_V into formula (11) for one year of the historical data which generates historical asset market values V_t. The volatility of these values produces new σ_V estimate; one should thus iterate until σ_V converges, meaning that the absolute difference between the past two iterations of σ_V does not exceed the limit of 10^{-3}. Finally, the calibrated σ_V is used to compute V_t for the past year and based on that one can also compute the log returns on the market asset value μ_V.

The theoretical and practical Merton versions deviate also in the computation of the distance to default (d_2). As opposed to the theoretical Merton model under the risk-neutral measure, the Merton DD model computes d_2 using an estimate of the expected asset's annualized returns μ_V instead of a risk-free interest rate. This is illustrated in equation (18).

$$d_2 = \frac{\ln\left(\frac{V_t}{F}\right) + \left(\mu_V - \frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V\sqrt{(T-t)}} \tag{18}$$

In a similar fashion as in the theoretical model, in order to produce the probability of default π^{Merton}, one has to utilize the formula (14). By doing so, the normal distribution, which converts distances to default d_2 into the PD, is assumed (Bharath & Shumway, 2008).

1.1.3 A Naïve Alternative Merton Model

Bharath and Shumway (2008) questioned whether the rigorous asset volatility calculation procedure, of which consists the Merton DD model, really adds to the PD estimation accuracy. As a result, they produced a Naïve alternative predictor that is not based on a complicated derivation or procedure. In the contrary, the objective of this model is primarily to test whether the time consuming algorithm presented in subsection 1.1.2 is worth implementing at all.

One can assess it by using the same information as the Merton DD model does and similarly, by keeping its functional structure which is reflected in the equations (14) and (18). At the same time, the goal is to prove that it performs as well as a simple model can in order to stand a chance of rejecting the hypothesis that the Merton DD model is superior. In case of rejection, one can also neglect the statement that the calibration technique adds value to the PD estimation (Bharath & Shumway, 2008).

Bharath and Shumway (2008) use the same estimate of the market value of debt as the Merton DD model, which is reflected in the equation (15). Nonetheless, they substantially simplify computation of the market asset value as they simply sum up the estimated market value of debt and the observed market capitalization for the purposes of approximation. Additionally, they approximate a firm's stock return in the previous year r_{it-1} as the expected return on its assets as presented in equation (19).

$$\mu_{V,Naive} = r_{it-1} \tag{19}$$

Moreover, Bharath and Shumway (2008) noticed that firms closer to default have riskier debt as well, which is generally positively correlated with the riskiness of equity. In addition, they set a lower boundary of volatility, namely to 5 %. As a result, their approximated volatility of debt results in:

$$\sigma_{D,Naive} = 0.05 + 0.25\sigma_E \tag{20}$$

As the estimates of debt's and equity's volatilities are known, they use the weighted average of the two as the volatility of assets:

$$\sigma_{V,Naive} = \frac{E}{E+F}\sigma_E + \frac{F}{E+F}\sigma_{D,naive} \tag{21}$$

All in all, Bharath and Shumway (2008) generally rely on the same set of data as the Merton DD model. However, the main difference lies in the simplicity of the Naïve model and the straightforwardness of its implementation.

Finally, the Naïve alternative attempts to keep the functional structure of the Merton model. As a result, the probability of default π^{Naive} is calculated by inserting the naïve estimates, namely asset's volatility and expected return, and the book value of assets, in the formulas (18) and (14).

1.2 Black-Cox Model

Soon after the Merton model was revealed, the first important extension of the basic model was introduced by Fisher Black and John C. Cox in 1976. The Black-Cox model (also BC model) is the first so-called first passage model type out of many that followed. The main characteristic of these kind of structural credit risk models is that they relax the assumption 8 in the Merton model which does not reflect reality well enough (Elizalde, 2006).

The Black-Cox model keeps all Merton's assumptions with one exception – instead of the Merton's assumption 8 it assumes that the default of a firm can occur anytime until the maturity of a firm's obligations. Such modelling results in the default time being uncertain (Sundaresan, 2013). Hence, a firm is forced into default at the first point in time u when the firm's assets value falls below the deterministic safety covenant level $K(u)$. Originally, Black and Cox (1976) modelled the barrier using a classical exponential form $K(u) = Ke^{-\gamma(T-u)}$ for $K > 0$.

Consequently, debtholders force a firm to go into a default at time u as soon as the market value of assets drops below the safety covenant. This can be expressed as:

$$V_u \leq K(u) = K * e^{-\gamma(T-u)}, \forall s < u: V_s > K(s) = K * e^{-\gamma(T-s)} \tag{22}$$

Under these assumptions, the equity of a firm can no longer be considered as the European call option, however, it could be modelled as a continuously monitored down-and-out call barrier option. The payoff of our firm at the maturity T in the Black-Cox model equals (Davis, 2006):

$$E(T, V_T) = \max(V_T - F, 0) * 1_{M_T > 0} \tag{23}$$

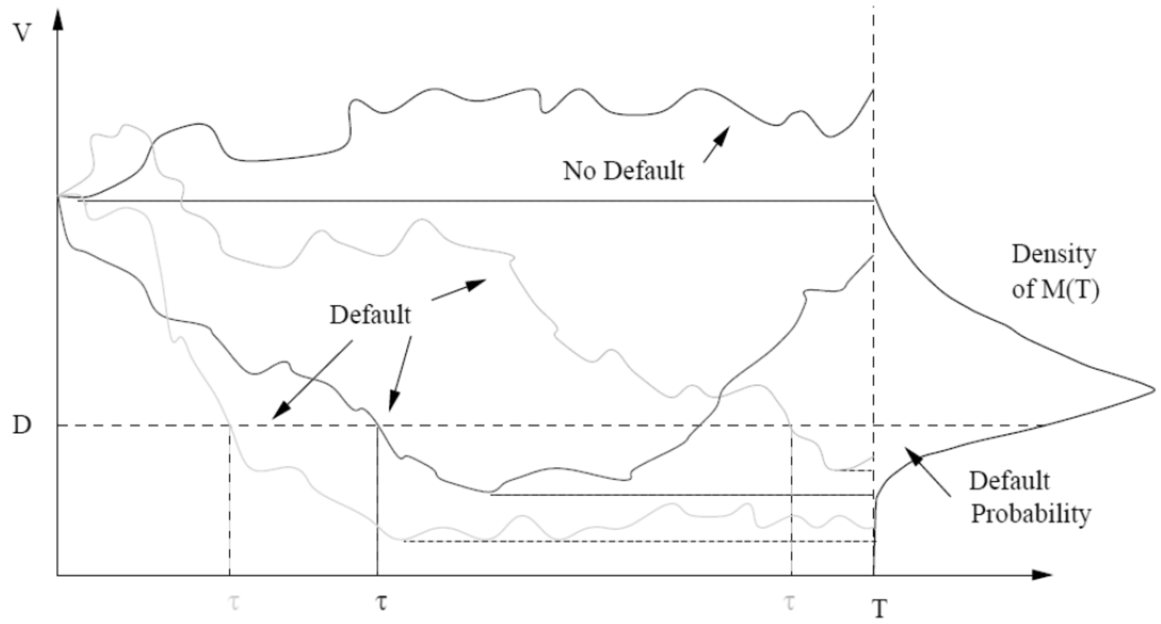
where:

- $1(\cdot)$ represents the indicator function;
- M_t is a process which stands for the minimal level of a difference between the assets and the safety covenant since the valuation time 0. Therefore:

$$M_t = \min_{0 \leq u \leq t} (V_u - K(u)) \tag{24}$$

The option pricing idea used in the Black-Cox model is displayed in Figure 2 where the special case of the constant barrier is shown. A firm goes into default for the three asset market value trajectories that fall below the obligations level D before the end of the forecasting period T, while it remains solvent in the one remaining case. It is noteworthy to mention that even though the trajectory recovers, hence market value of assets again exceeds the barrier, the Black-Cox model ignores the rest of the path as soon as the barrier is hit. Indeed, this is the main difference with respect to the Merton model, where the barrier is not considered.

Figure 2: Graphical presentation of asset market value trajectories in Black-Cox model with constant threshold D and forecasting period T.

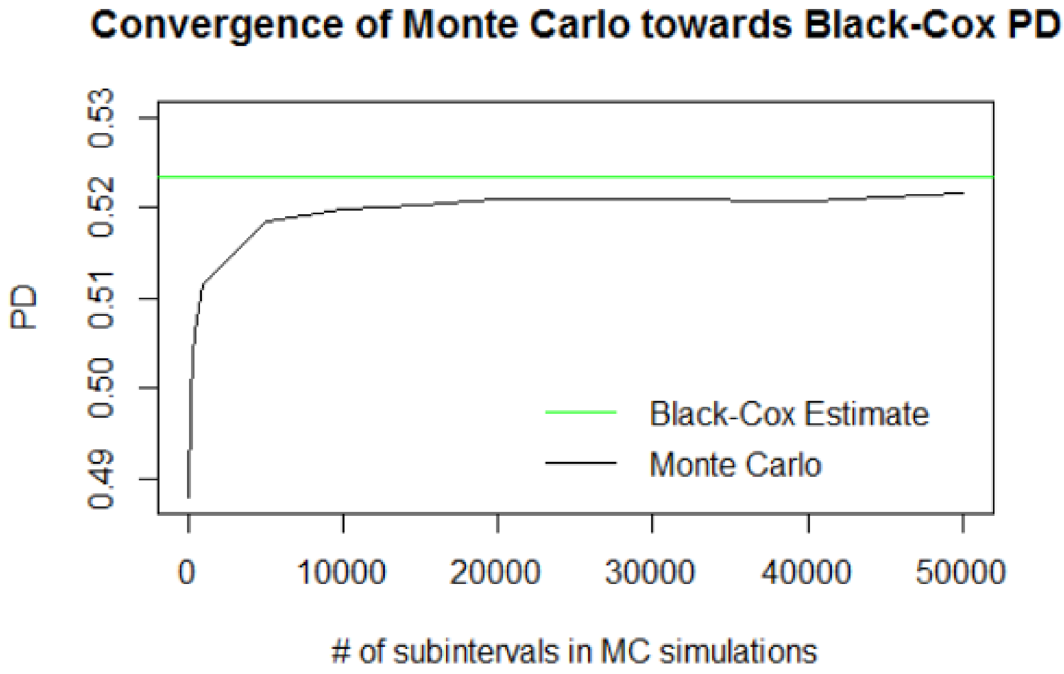


Source: Mišankova, Kočišova and Klištik (2014), p. 287.

Metwally and Atiya (2002) point out that the idea of a discretely monitored down-and-out barrier option might be more relevant in practice than the continuous one used in the Black-Cox model. Indeed, in practice the default does not happen exactly in the moment when assets hit the barrier and firms could still operate despite being insolvent for some time. One way of tackling it could be Monte Carlo simulations which may be time consuming. This problem is examined by Broadie, Glasserman and Kou (1997) who derive a correction factor for continuously monitored barrier option pricing. As a result, they obtain a more desirable approximation of a discretely monitored barrier option price. The idea behind it is that if a firm in the first-time passage models is monitored discretely, such firm may have hit the barrier if the frequency of monitoring would be continuous. However, the default between two consecutive monitoring times may be overlooked in case of discrete monitoring. Hence, the larger is the length of sub periods in Monte Carlo simulations, the larger will be the difference between the Monte Carlo and the analytical Black-Cox PD. Consequently, the latter PD estimate may overestimate the probability of default.

I show the difference in estimation between the two approaches as of March 2, 2009 in Figure 3, where an example of General Electric (GE) is considered. The number of simulations used for each tenor is 500,000, whereas the number of subintervals presented on the x axis ranges between 100 and 50,000. As intuition suggests, one can see that the increasing number of subintervals in Monte Carlo simulations indeed increases the estimated PD and finally converges towards the BC model estimate. Hence, the continuous framework of the Black-Cox model might generally slightly overestimate the PD. In Appendix 8 I present the R code used to produce the Monte Carlo estimates.

Figure 3: Convergence of PD obtained through Monte Carlo simulations (black line) towards analytical Black-Cox estimate (green line) by increasing the number of subintervals in Monte Carlo simulations.



Source: own work.

On the other hand, one should note that the Merton model is not affected by this fact as only the last observation counts in Merton's framework. Therefore, it does not matter whether paths were generated discretely or continuously for this basic structural credit risk model. Nonetheless, for the purposes of empirical comparison between the models I will not take into account the Monte Carlo simulations and the correction factors as these exceed the purpose of this paper. Moreover, such modelling would raise further questions related to what monitoring frequency is the most realistic.

The Black-Cox model inherits the Merton model's stochastic differential equation (8), which describes the equity's dynamics. However, in the Black and Cox (1976) model there is an additional boundary condition:

$$E\left(t, K * e^{-\gamma(T-t)}\right) = 0 \tag{25}$$

This condition introduces an additional restriction for the firm value dynamics, which clearly results in a fact that the probability of default will always be higher in the BC model than in Merton's. Black and Cox (1976) present the analytical formula (26) under the risk-neutral measure for the PD of a firm between time t till time $T > t$ which equals:

$$\begin{aligned} &PD(t,T) \\ &= 1 - N\left(\frac{\ln\left(\frac{V_t}{F}\right) + \left(r - \frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V\sqrt{(T-t)}}\right) \\ &+ \left(\frac{V_t}{K(t)}\right)^{1-\frac{2(r-\gamma)}{\sigma^2}} N\left(\frac{2 * \ln(K(t) - \ln(V_t F) + \left(r - \frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V\sqrt{(T-t)}}\right) \end{aligned} \tag{26}$$

for every t when $V_t > K(t)$. If this condition is not fulfilled, the boundary is already breached, and as a result, the company is going into default with the probability 1 (Mišankova, Kočišova & Klištik, 2014).

In the original Black-Cox (1976) paper, the parameter K is very general and could be chosen by the modeler. Nonetheless, many researchers including Sundaresan (2013) and Mišankova, Kočišova and Klištik (2014) opted for the parameter K to equal the face value of debt F. In the upcoming sections I follow the same approach. In a similar fashion, the choice of parameter γ, which affects the barrier up until the maturity, is arbitrary. Elizade (2008) ignores this parameter and sets it to 0, which results in a constant threshold. On the other hand, the same author points out that setting it to risk-free rate r could be meaningful choice too. By doing so, the threshold would represent the discounted face value of debt. In the practical section 3 I analyze both suggested approaches within which the PD estimates are denoted as π^{BC} and $\pi^{BC}_{Cons\ T}$, respectively.

1.3 Longstaff-Schwartz Model

In 1995 Francis A. Longstaff and Eduardo S. Schwartz developed an important extension of the aforementioned models, a two-factor structural credit risk model. They observed that the more basic Merton and BC models tackle only the default risk using a Black-Scholes framework but they never take into account the interest rate risk. The reason for it is that the previous two covered models assume rates to be constant which is an oversimplification and in general not realistic (Longstaff & Schwartz, 1995; Tarashev, 2008). Longstaff and Schwartz (1995) even observe that credit spreads are generally negatively correlated with respect to interest rates, which is not incorporated in any of the early structural credit risk models.

The starting point of the Longstaff-Schartz model (also the LS model) is a special case of the BC model. Longstaff and Schwarz (1995) set the Black-Cox model's parameter γ to 0, consequently building on a model with a constant barrier. Davidenko (2007) argues that the threshold is thus set to 100 % of the face value of debt which is not an indisputable modelling choice. Based on his analysis, he empirically derived that the default boundary should instead be close to 72 % in order to obtain more optimal results. The economic rationale for it is that a company could temporarily survive in spite of being insolvent. Nevertheless, as this is not the original LS model, I will not adjust the default threshold in the practical part of the thesis.

Longstaff and Schwarz (1995) keep all Black-Cox model's assumptions except Merton's assumption number 6, which is relaxed. Hence, they allow risk-free rate to be stochastic and correlated with the dynamics of a firm's asset returns. In general, this correlation between interest rates and asset returns is supposed to be negative, however, it is not a restriction as per the model construction (Tarashev, 2008).

Stochastically modelled short-term risk-free interest rate in LS model is inspired by a research paper of the Czech mathematician Oldrich Vasicek. The Vasicek (1977) model presents one of the first widely known and used stochastic interest rate models. The main goal of the model is to generate a simple closed-form solution for the risky debt pricing. At the same time, it produces an approximate solution for the probability of default. Even though it cannot provide a close-form formula, the numerical solution for the PD based on a formula is an exception rather than a rule within the two-factor modelling framework (Collin-Dufresne & Goldstein, 2001).

The biggest drawback of Vasicek's mean reversion short-rate model is that it can produce negative interest rates with a positive probability. That said, nowadays one can observe that slightly negative risk-free interest rates in European markets are in fact not completely unlikely. Nonetheless, Longstaff and Schwartz (1995) justified their modelling choice by a low probability that negative interest rates occur whenever using reasonable parameters. The modelling of interest rates is graphically displayed in Figure 4 where the mean reversion idea is illustrated.

In the LS model, the dynamics of the short-term risk-free interest rate *r* in time *t* is thus given by the following Vasicek's stochastic differential equation (27):

$$dr_t = (\alpha - \beta r_t)dt + \eta dW_{r,t} \tag{27}$$

Figure 4: Mean reversion of short rate in Vasicek model

Source: Burgess (2014), p. 3.

In the above formula (27) the constant terms are assumed to be strictly positive and they can be described as follows (Tarashev, 2008):

- η is the volatility of a short-term interest rate.
- β reflects a mean reversion speed.
- $\theta = \frac{\alpha}{\beta}$ represents the expected long-term average of r_t and can be considered as a mean reversion level as well.

On the other hand, asset value dynamics remains identical to the Merton and the BC models (Longstaff & Schwartz, 1995):

$$dV_t = r_t V_t dt + \sigma_V V_t dW_t \qquad (28)$$

In above equations both W_t and $W_{r,t}$ represent standard Brownian motion processes, however, the LS model allows them to be correlated. The correlation between the two in Longstaff and Schwartz (1995) is given by:

$$Corr\left(W_t, W_{r,t}\right) = \rho \qquad (29)$$

Generally, one expects ρ to be negative. Assuming such setting, the corollary of a rise in interest rates will be a negative impact on the market asset value and therefore, the probability of default will be increased due to this correlation (Tarashev, 2008).

The payoff in the LS model remains the same as in the BC model with a constant barrier level. However, the Longstaff and Schwartz (1995) model is a two-factor model where the equity level is a function $E = E(t, r_t, V_t)$ and where a stochastic differential equation turns to:

$$\frac{\sigma_V^2}{2} V^2 E_{VV} + \rho \sigma_V \eta E_{rV} + \frac{\eta^2}{2} E_{rr} + r_t V_t + (\alpha + IRR - \beta r_t) E_r - r_t E + E_t = 0 \qquad (30)$$

Let me emphasize that the parameter IRR represents a constant which reflects the interest rate risk. Nevertheless, for the sake of simplicity, I will assume it to be set to 0.

1.3.1 Calibration of Vasicek Parameters

Generally, there exist two approaches regarding how to fit the parameters of an interest rate model such as Vasicek. These two ways fundamentally differ in the data being used in order to provide an estimate of parameters. One the one hand, the first approach is based on the historical estimation that obtains parameters directly from the past time series of an interest rate data. When employing this methodology, the time window used for an estimation is a very important modelling choice. However, especially in the stressed periods this approach leads to unstable estimations as past time series may be difficult to fit with reasonable parameters (Burgess, 2014; Cuchiero, 2006).

The alternative approach is the calibration of parameters to the current market data rates, which is more desirable in practice. This is due to the general aim that the Vasicek parameters reflect the present rather than the past and so the assumption that history will repeat itself can be relaxed. Generally, the Vasicek parameters could also be calibrated to fit derivatives such as caps, floors, swaptions or volatility structure of these products. Nonetheless, less technical and the most common approach for theoretical purposes is to calibrate the parameters according to the current yield curve (Cuchiero, 2006). Due to the simplicity and availability of the data, I will only consider the latter calibration which has also been used by Eom, Helwege and Huang (2004).

The market Treasury yield data has been obtained from the Federal Reserve Bank of St. Louis (2017) website. The data covers market yields $R_M(0, T)$ with the following maturities T, which are expressed in years. Nonetheless, note that the data for 20- and 30-year maturities is missing for several years between 1990 and 2006.

$$T = \frac{1}{4}, \frac{1}{2}, 1, 2, 3, 5, 7, 10, 20, 30 \tag{31}$$

On the other hand, the theoretical yield will be a function of maturity and denoted as $R(t,T)$. As Vasicek (1977) derived a bond price in a closed-form formula, there also exist a closed-form formula for the theoretical yield.

$$R(t,T) = -\frac{1}{T} P\big(t, t+T, r(t)\big) \tag{32}$$

Consequently, the theoretical yield in the Vasicek (1977) model can be computed through equation (33) given the parameters α, β and η, and the short rate r_0.

$$R(0,T) = \left(\frac{\alpha}{\beta} - \frac{\eta^2}{2\beta^2}\right) + \left(r_0 - \frac{\alpha}{\beta} + \frac{\eta^2}{2\beta^2}\right)\frac{1 - e^{-\beta T}}{\beta T} + \frac{\eta^2}{4\beta^3 T}\left(1 - e^{-\beta T}\right)^2 \tag{33}$$

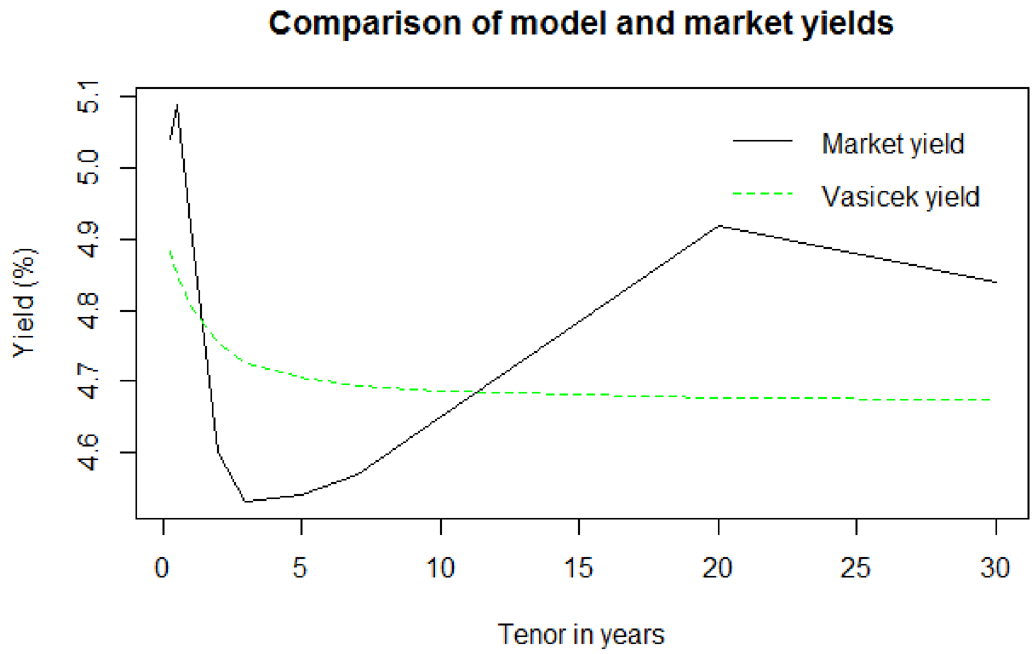
I follow Cuchiero (2006) and Eom, Helwege and Huang (2004), employing the least-squares approach in order to fit the Vasicek parameters to the market yield as well as possible. More specifically, the aim is to minimize the deviations between the two for all the observed tenors T. This problem can be expressed as:

$$\min_{\alpha,\beta,\eta}\big(R_M(0,T) - R(0,T)\big)'\,\big(R_M(0,T) - R(0,T)\big) \tag{34}$$

As opposed to Cuchiero (2006), who relied on the Matlab "lsqnonlin" function, I tackled this optimization problem in R by utilizing the function "nlminb" within the "optimx" package developed by Nash, Varadhan and Grothendieck (2015). This optimization function uses the quasi-Newton method and allows boundary constrains, which are needed in our calibration (Nash, 2014). In particular, by the definition, the η and β have to be larger or equal to 0 since they stand for the volatility of the rate and speed of mean reversion, respectively.

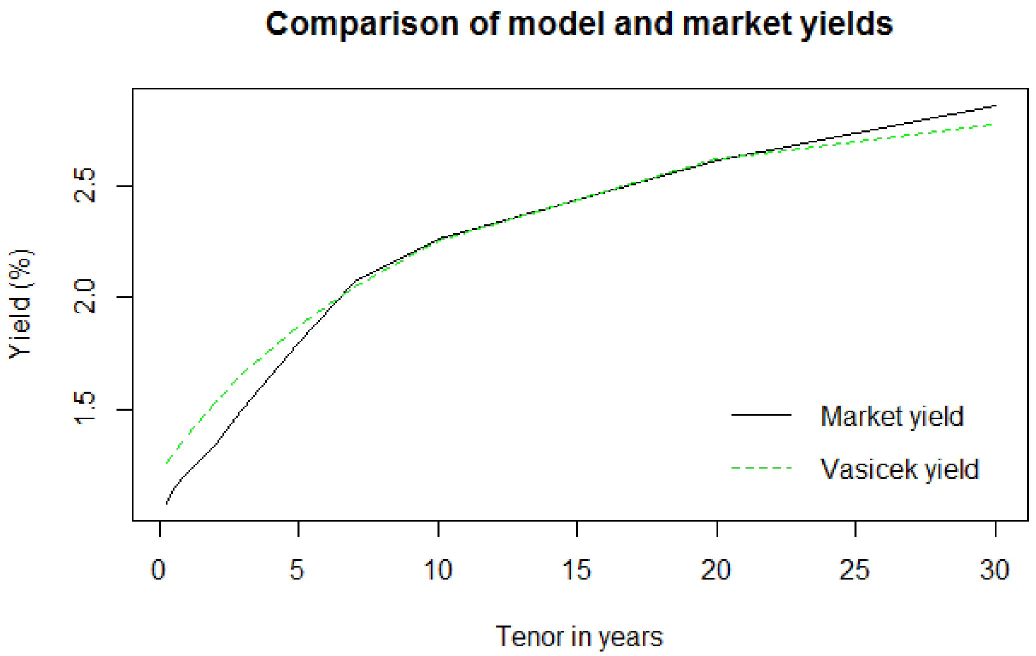
In practice it is not uncommon for the calibration of Vasicek parameters to give unreasonable estimates when trying to come up with parameters that fit more complex yield curve shapes that can be observed in the market. Keller-Ressel and Steiner (2007) claim that it is common to blame the low number of parameters in the model which affects the flexibility of the model when fitting the data. As a result, the model cannot capture the shapes of yield curve such as curve with a local minimum, which may happen during stressed periods or when expecting them (see Figure 5). On the other hand, in Figure 6 I display that the model can perform satisfactorily when the yield curve is rather smooth.

Figure 5: Comparison of calibrated Vasicek yield (green dashed curve) and market yield (black curve) curves as of April 2, 2007.



Source: own work.

Figure 6: Comparison of calibrated Vasicek yield (green dashed curve) and market yield (black curve) curves as of August 1, 2017.



Source: own work.

Thus, it makes sense to add some additional boundary conditions in order to prevent these scenarios in a similar fashion as Cuchiero (2006). These are arbitrary and strictly a modelling choice. In particular, I applied additional conditions, which impose β not to exceed 5 whereas the long-term average θ should be lower than 10 %. These can be considered extreme values. Apart from that, it turns out that in the times when economy is rather stable, the estimates for short-rate volatility tend to be very close to 0 and explode in the stressed conditions. Thus, I limited η between $\frac{1}{2}s_r \leq \eta \leq \frac{3}{2}s_r$ where s_r stands for an unbiased estimate of the historical standard deviation over the past 1-year window. As a result, the Vasicek model produces reasonable parameters.

Let me mention that I defined 1 year with the BUS/252 day count convention described in OpenGamma (2013). Here one counts the fraction of the year between the two dates by counting business days as specified in the equation (35). Note that this choice is arbitrary as any other standard convention could be used instead.

$$year\ Fraction = Business\ days/252 \tag{35}$$

1.3.2 Implementation of Longstaff-Schwartz Model

Longstaff and Schwartz (1995) proposed a numerical solution for the probability of default estimation under the T-forward measure. Dynamics of a risk-free rate and the log market asset value under T-forward measure are presented in Appendix 2.

The Longstaff and Schwartz (1995) PD approximation is a limit of the expression (36) when the number of simulations n is sent towards infinity. The algorithm is illustrated with the following equations:

$$\lim_{n\to\infty} Q(X, r_0, T, n) = \sum_{i=1}^{n} q_i \tag{36}$$

$$q_1 = N(a_1) \tag{37}$$

$$q_i = N(a_i) - \sum_{j=1}^{i-1} q_j N\left(b_{ij}\right); i > 1 \tag{38}$$

$$a_i = \frac{-lnX - M\left(\frac{iT}{n}, T\right)}{\sqrt{S\left(\frac{iT}{n}\right)}} \tag{39}$$

$$b_{ij} = \frac{M\left(\frac{jT}{n}, T\right) - M\left(\frac{iT}{n}, T\right)}{\sqrt{S\left(\frac{iT}{n}\right) - S\left(\frac{jT}{n}\right)}} \tag{40}$$

where X represents the ratio between V_0 and K and has to be above 1. That said, if the value of assets already is below the threshold, the company defaults with the probability of 1 as per construction of the model. Furthermore, the functions M and S are defined as:

$$\begin{aligned} M(t,T) = {} & \left(\frac{\alpha - \rho\sigma_V\eta}{\beta} - \frac{\eta^2}{\beta^2} - \frac{\sigma_V^2}{2}\right)t + \left(\frac{\rho\sigma_V\eta}{\beta^2} + \frac{\eta^2}{2\beta^3}\right)e^{-\beta T}\left(e^{\beta t} - 1\right) \\ & + \left(\frac{r_0}{\beta} - \frac{\alpha}{\beta^2} + \frac{\eta^2}{\beta^3}\right)\left(1 - e^{-\beta t}\right) - \left(\frac{\eta^2}{2\beta^3}\right)e^{-\beta T}\left(1 - e^{-\beta t}\right) \end{aligned} \tag{41}$$

$$\begin{aligned} S(t) = {} & \left(\frac{\rho\sigma_V\eta}{\beta} + \frac{\eta^2}{\beta^2} + \sigma_V^2\right)t - \left(\frac{\rho\sigma_V\eta}{\beta^2} + \frac{2\eta^2}{\beta^3}\right)\left(1 - e^{-\beta t}\right) \\ & + \left(\frac{\eta^2}{2\beta^3}\right)\left(1 - e^{-2\beta t}\right) \end{aligned} \tag{42}$$

Collin-Dufresne and Goldstein (2001) argue that the above formula serves only as an approximation of the true solution of the LS model. Furthermore, Huang and Zhou (2008) claim that the PD in the LS model should rather be solved by making use of numerical methods, whereas Collin-Dufresne and Goldstein (2001) proposed their exact solution. Nonetheless, for purposes of this thesis I implement the original procedure. I opted for $n = 5000$, which enables to calculate a PD for each time observation of each company in a reasonable time for all observations. The implementation of the model in R is exhibited in Appendix 5.

1.4 Collin-Dufresne and Goldstein Model

Pierre Collin-Dufresne and Robert S. Goldstein aimed to develop a structural credit risk model which would generate increased credit spreads for firms with lower leverage. In addition, they wanted to achieve that it would be less sensitive with respect to the changes in asset levels which would have been in line with observations in empirical studies. As a result, Collin-Dufresne and Goldstein (2001) built a first-passage model type in which asset and debt values are generally correlated. As observed in practice, when obligations of a firm increase, the asset side of the balance sheet in principle gets higher as well and consequently, the leverage ratio appears to be both mean-reverting and stationary.

Even though the added value of the Collin-Dufresne and Goldstein model (also the CDG model) comes from the modelling of the mean-reverting leverage ratio between assets and obligations, the base of the model consists of stochastic interest rates. Stochastic modelling of interest rates is a particularly important element as firms have a tendency to issue more debt when interest rates decrease and vice versa. Thus, the CDG model incorporates this idea through a correction factor for the level of debt which takes into account how far away the current risk-free rate is from its long-term level. This has not been observed in structural credit risk modelling before and such model better reflects the reality of the leverage ratio being stationary (Collin-Dufresne & Goldstein, 2001).

The CDG model retains same assumptions as the aforementioned LS model, meaning all Merton's assumptions except the assumptions 6 and 8. Moreover, the assumption 6 is even further relaxed in comparison to the LS model. Thus, the default event appears whenever the assets level drops below the stochastic barrier, a face value of debt, which can even be correlated with firm's assets (Collin-Dufresne & Goldstein, 2001).

First of all, for the sake of convenience, the log-process of asset dynamics $y_t = logV_t$ is defined. Based on that, under the risk-neutral measure it follows that:

$$dy_t = \left(r_t - \frac{\sigma_V^2}{2}\right)dt + \sigma_V dW_t \tag{43}$$

As already introduced by Longstaff and Schwartz (1995), the CDG model takes into account the potential correlation ρ between a market asset return and a risk-free rate, which is already defined with the equation (29). In addition, it models the interest rate's mean reversion in the identical way as the LS model, through the Vasicek model presented in (27).

Nevertheless, as opposed to the constant default limit suggested by the LS model, Collin-Dufresne and Goldstein (2001) introduced a log-default barrier k_t, which is defined by the following formula (44). It depends both on the asset return's and interest rate's dynamics, and results in a stationary leverage ratio.

$$dk_t = \lambda(y_t - \nu - \phi(r_t - \theta) - k_t)dt \tag{44}$$

Firstly, λ represents the impact of the difference between $y_t - \nu - \phi(r_t - \theta)$ and k_t on the rise of log debt. Whenever the log debt k_t drops below the log assets y_t adjusted for a constant term ν and an interest rate effect $\phi(r_t - \theta)$, the firm starts to increase it and vice-versa. One can notice that by setting λ to 0, the CDG model simplifies into the LS model.

Secondly, the parameter ϕ reflects the leverage ratio sensitivity with respect to the risk-free interest rate and is assumed to be non-negative, which results in high risk-free interest rates causing a lower level of debt issuing. Indeed, this is consistent with the observation in practice. θ can be interpreted as an interest rate's long-term average as in section 1.3, while ν stands for the target log leverage ratio given that the interest rates are close to their long-term average.

1.4.1 Implementation of Collin-Dufresne Model

In order to estimate the CDG parameters λ, ν and ϕ researchers use different approaches. Eom, Helwege and Huang (2004) estimated them by regressing the leverage ratio on the lagged ratio and the risk-free rate. On the other hand, Eisenthal, Feldhutter and Vig (2017) used the fixed CDG parameters, identical to the ones proposed by the original Collin-Dufresne and Goldstein (2001) paper. I follow constant parameters approach by setting barrier parameters to $\lambda = 0.18$, $\nu = 0.6$, and $\phi = 2.8$ and I test the sensitivity of the PD estimates with respect to these parameters in section 3.1. On the other hand, the estimation of Vasicek parameters is identical to the one presented in section 1.3.1.

In 1995 Longstaff and Schwartz proposed their approximation for the PD calculation. In a similar manner Suo and Wang (2006), and Eom, Helwege and Huang (2004) implemented the CDG model with numerical solutions for both risky bond pricing and the PD. They discretized time to n subintervals and claimed that by sending this number of subintervals towards infinity, the estimated PD converges towards the approximated solution. However, it should be noted that they simplified the original exact implementation proposed in Collin-Dufresne and Goldstein (2001) that discretizes the interest rate space, too. In particular, Collin-Dufresne and Goldstein (2001) take into account the fact that their model is not a single-factor Markov process. Due to it, they discretize the risk-free rate space, hence, their approach represents a better approximation of the true solution (Collin-Dufresne and Goldstein, 2001).

As the probability of default derivation for the CDG model is rather tedious and out of the scope of this thesis, I only present the final solution in this subsection. Firstly, it is convenient to define a log-leverage ratio as follows.

$$l_t = k_t - y_t \tag{45}$$

A sum of discretized probabilities under the T-forward measure, which converges towards the exact solution, is defined with the following equations (Collin-Dufresne & Goldstein, 2001):

$$Q^T(r_0, l_0, T) = \sum_{j=1}^{n_T} \sum_{i=1}^{n_r} q(r_i, t_j) \tag{46}$$

$$q(r_i, t_1) = \Delta r \Psi(r_i, t_1) \tag{47}$$

$$q(r_i, t_j) = \Delta r \left(\Psi(r_i, t_j) - \sum_{v=1}^{j-1} \sum_{u=1}^{n_r} q(r_u, t_v) \psi(r_i, t_j | r_u, t_v) \right) \tag{48}$$

$$\Delta r = \frac{\bar{r} - \underline{r}}{n_r} \tag{49}$$

where:

- n_T represents the number of identical time subintervals that divide interval up to the maturity; and
- n_r stands for the number of intervals that discretize the r-space into equal intervals.

$\underline{r}$ and $\bar{r}$ are lower and upper rate boundaries, respectively, introduced as interest rates are assumed to be stochastic in the model. For the sake of faster computations, one should not account for all scenarios but rather opt for the limits that are statistically unlikely to be reached. Generally, these values are arbitrary, however, Collin-Dufresne and Goldstein (2001) set them at three standard deviations below and above the risk free-rate's long-term average θ. Therefore:

$$\underline{r} = \theta - 3\sqrt{\frac{\eta^2}{2\beta}} \tag{50}$$

$$\bar{r} = \theta + 3\sqrt{\frac{\eta^2}{2\beta}} \tag{51}$$

The following set of equations leads towards the final solution:

$$\Psi(r_t, t) = \pi(r_t, t | r_0, 0) N\left(\frac{E_0^T(l_t | r_t, l_0, r_0)}{\sqrt{Var_0^T(l_t | r_t, l_0, r_0)}} \right) \tag{52}$$

$$\psi(r_t, t|r_s, s) = \pi(r_t, t|r_s, s)N\left(\frac{E_s^T(l_t|r_t, l_s = 0, r_s)}{\sqrt{Var_s(l_t|r_t, l_s = 0, r_s)}}\right) \tag{53}$$

$$E_s(l_t|r_t, l_s, r_s) = E_s(l_t) + \frac{Cov_s^T(l_t, r_t)^2}{Var_s^T(r_t)}\big(r_t - E_s(r_t)\big) \tag{54}$$

$$Var_s(l_t|r_t, l_s, r_s) = Var_s^T(l_t) - \frac{Cov_s^T(l_t, r_t)^2}{Var_s^T(r_t)} \tag{55}$$

where π is the transition density function for the risk-free interest rate in the Vasicek model (Collin-Dufresne & Goldstein, 2001). The transition density takes the Gaussian form and is defined as (Ait-Sahalia, 1999):

$$\pi(r_t, t|r_s, s) = \frac{1}{\sqrt{2\pi}\gamma}e^{-\frac{\left(r_t-\theta-(r_s-\theta)e^{-\beta(t-s)}\right)^2}{2\gamma^2}} \tag{56}$$

with

$$\gamma^2 = Var(r_t|r_s) = \frac{\eta^2}{2\beta}(1 - e^{-2\beta(t-s)}) \tag{57}$$

The final solutions for conditional moments of processes l_t and r_t in the CDG model observed in the equations (52) – (55) are derived in details in Collin-Dufresne and Goldstein (2001). As their derivation is out of the thesis' scope, the final solutions can be found in Appendix 3.

One should be aware that the Collin-Dufresne and Goldstein (2001) solution only converges towards the exact PD. However, since the algorithm is quite time consuming, practitioners may be forced to choose low n_r and n_T. In particular, for the purposes of the thesis I set $n_r = 50$ and $n_T = 50$. As a result, it can occur that some estimated PDs lie outside of the expected interval $[0, 1]$. By further increasing n_r and n_T, this issue vanishes away. As a result, as it can be observed in the R code in Appendix 4 where CDG model is implemented, a sanity check is advised in order to assure that these limits are not exceeded.

1.5 Cox Proportional Hazard Model

For the purposes of answering the research questions of my thesis, a method that enables a comparison between the examined structural credit risk models is needed. Therefore, I utilize the Cox proportional hazard regression model as the main tool when assessing the sufficiency and importance of structural credit risk models discussed earlier in this section.

The problem of forecasting a firm's time to default is very similar to forecasting the time until events such as death occurs. The latter example is where the name of the survival analysis origins from, even though it is often employed to model financial events as well. In particular to the credit risk modelling, this model deals with the relationship between the time to default and its predictors (Fox, 2002; Bharath & Shumway, 2008). An important feature when modelling the time to default, or more generally speaking, the survival time, is the fact that several companies' default have not happened yet and thus, the true time to default is still unknown. The lack of this information is described as the right-censored data since one only knows for sure that the time to default is beyond the last observation in our data set (Lin, 2003).

The basic concept in the survival analysis is to model the survival time T as a random variable. Its distribution could be defined as the survival function:

$$S(t) = P(T > t) = 1 - F(t); t < T \tag{58}$$

for some future point in time t. On the other hand, F denotes the cumulative distribution function of T (Fox, 2002).

Nonetheless, due to the censorship, according to Lin (2003), it is preferable to present the survival time through the hazard function. A hazard is defined as an instantaneous probability of default at the time t among those firms that have not defaulted yet. If f is the density function of T and if the survival time T is continuous, then one can define the net hazard function as:

$$h(t) = \lim_{\Delta t \downarrow 0} \frac{P(t \leq T < t + \Delta t | T \geq t)}{\Delta t} = \frac{f(t)}{S(t)} \tag{59}$$

In order to involve the censorship, Lin (2003) defines U as the censorship time, which is the last available point in time where an observation still exists. Let me denote:

$$\delta = I(T \leq U) \tag{60}$$

where $\delta = 0$ associates to all censored observations in the data. Moreover:

$$X = \min(T, U) \tag{61}$$

Thus, for every t, where $P(X > t) > 0$, the crude hazard rate can be given by:

$$h^{\#}(t) = \lim_{\Delta t \downarrow 0} \frac{P(t \leq T < t + \Delta t | T \geq t, U \geq t)}{\Delta t} \tag{62}$$

In survival analysis the crucial assumption being made is that the net and crude hazard rates presented in the formulas (59) and (62), respectively, are matched for every t in such a manner that $P(X > t) > 0$. This assumption is generally fulfilled with the independence among T and U, which implies independent censoring (Lin, 2003). Hazard rates could be modelled in a log-linear type of a model as:

$$\log h_i(t) = \alpha + x_i(t)'\beta \tag{63}$$

where i represents the observations and x accounts for the explanatory variables. Besides, α is a parameter that could be interpreted as a log-baseline hazard or a hazard rate when all covariates are set to 0. As observed, this is a parametric model as the baseline hazard function is imposed in a constant way (Fox, 2002).

On the other hand, one of the most widely used methods in the survival analysis is the Cox proportional hazard model (Cox PHM**),** which is employed by Bharath and Shumway (2008) for instance. As opposed to the linear model, the Cox PHM does not pre-specify the form of the baseline hazard function and thus, the hazard rate is modelled as:

$$\log h_i(t) = \alpha(t) + x_i(t)'\beta \tag{64}$$

Thus, the logarithms of explanatory variables x have a multiplicative effect on the hazard rate. These vary in time and among individuals as well. On the contrary, $\alpha(t) = log(h_0(t))$ is a function of time but constant for every individual i. As a result, the Cox PHM does not impose the risk of default to be constant but allows it to fluctuate in time.

Furthermore, because the baseline hazard function not being specified, the constant intercept term is redundant as it is already incorporated in the function $h_0(t)$. Since the Cox proportional hazard model considers $h_0(t)$ to be a general and unspecified function, the Cox PHM is known as a semi-parametric model. Nevertheless, the model assumes the hazard rate to be positive, which is imposed by the hazard function's definition in equation (59) (Fox, 2002; Lin, 2003; Thomas & Reyes, 2014).

The estimation of parameters β in the Cox PHM, presented in the equation (64), is usually solved by the maximum likelihood estimation (MLE) approach under the assumption of random censoring. The detailed discussion of the MLE estimation procedures can be found in Cox and Oakes (1984) and is beyond the scope of this research.

For the purposes of implementation of the Cox PHM model in statistical programming language R, I use the functions *survfit* and *coxph*. Thomas and Reyes (2014) claim that these produce the Cox PHM estimates even in the presence of time-varying coefficients as long as the data is correctly ordered. I exhibit practical implementation of the CPH regression model in Appendix 9.

1.5.1 Cluster Robust Variance Estimate

The structure of the data used in this thesis is a panel data as the PD is estimated for several companies on the monthly basis. Hence, time is an important component. Generally, the autocorrelation in the dataset can hardly be ignored when performing regressions, including the Cox proportional hazard regression. In order to produce robust standard errors, the efficient score residual clustered by companies is needed. The main consequence of this method is an adjustment of standard errors whereas the estimates are maintained. As a result, when it comes to the rejection or acceptance of the hypothesis, which is the main aim in this thesis, it is important to rely on a method that produces accurate results (StataCorp, 2017).

In general, the autocorrelated residuals produce a biased standard error estimation. Nonetheless, the cluster robust estimates heal this issue. In the statistical software package R, which I used for the purposes of analyzing the data within the scope of this thesis, the robust estimate of variance is accessible through the *coxph* function. To account for it, one can simply add the element *cluster* into the right side of the CPH regression which clusters based on the specified company IDs (Cameron & Miller, 2015; Therneau & Lumley, 2017).

The cluster robust estimation relaxes the assumption that the observations are independent. By using this technique, it is still assumed that the observations are independent across the companies whereas residuals can be correlated within the same cluster. For each CPH regression performed in section 3, I employed this type of estimation. The theory of techniques used for the cluster-robust variance estimating is above the scope of this thesis. However, its use is crucial for the validity of the CPH regressions in section 3.

1.6 Market Implied Default Probabilities

Apart from the structural credit risk models, discussed in my thesis, the PD of a particular country or company could also be extracted from the market instruments that embed this information in their price. Implying the PDs from the market data is valuable when comparing them against the PDs computed by the structural credit risk models presented in the earlier subsections. Furthermore, it may provide an additional insight when dealing with research questions such as whether structural credit risk models are sufficient or at least important predictors.

According to Bharat and Shumway (2008) there are many research papers that are attempting to explain the probabilities of default based on the corporate bond yield spreads. On the other hand, there is less literature looking at the relationship between PDs and credit default swap (CDS) prices as these instruments emerged only in the last two decades (Bharath & Shumway, 2008). Within the scope of this thesis, only the implied probabilities from the CDS instruments will be considered in order to challenge the hypothesis of structural credit risk models' importance.

The CDS represents a base in the market of credit derivatives as it is the most liquid product in the sector. This instrument is an agreement between two market counterparties which enables them to trade the credit risk of an issuer between each other. The issuer is in general a third reference entity, not necessarily related with any of the two investors. A credit default event includes bankruptcies, however, it could have a broader meaning, which would take into account failures of the issuer to pay back outstanding obligations. In specific cases of credit default swaps, the definition of the default event can also count restructuring of loans or bonds (Beinstein & Scott, 2006).

The CDS was constructed back in 1994 by JP Morgan Inc. in order to get rid of a credit risk exposure and to pass it to their counterparties. The popularity of credit derivatives, including the CDS, soon started to increase – before the financial crisis the total notional of credit derivatives has risen from 180 billion USD in 1997 to 2 trillion USD and more by 2002 (Bharath & Shumway, 2008; Augustin, Subrahmanyam & Tang, 2016). According to the European Commission (2016), the global amount of outstanding credit default swaps reached 12 trillion USD in 2016 and has been rapidly growing in the last few years, especially due to the lower volatility in the credit market. The peak was nonetheless reached in 2007 when the Global financial crisis kicked in (Wen & Kinsella, 2016).

The CDS is a financial product primarily meant to be used for the hedging purposes. Hence, the buyer of the CDS is considered to have bought insurance and holds a similar credit risk position as an investor that short sells a bond. Thus, this party transfers credit risk to the counterparty. On the other hand, the seller sells protection and receives similar benefits as when having a long position in a bond (Beinstein & Scott, 2006; Bharath & Shumway, 2008).

As a buyer enters the CDS, he is obliged to pay the selling party a predetermined premium s on a regular basis until the maturity of the credit default swap T, or until the potential default event of the issuer. The most typical maturity for a corporate CDS is 5 years, whereas it varies between 1–10 years for a sovereign CDS (Chan-Lau, 2006). Besides, a periodic premium could be paid quarterly, semiannually or annually and generally, buyer's fixed payments are paid in arrears. As a result, the final accrued payment has to be paid even upon the default of the reference entity. On the other hand, the seller in the CDS trade promises to buy back from his counterparty the bond of the defaulted issuer at its par value. Thus, in case of the default event, the payoff of the credit default swap's buyer equals one minus the recovery rate, if the notional is assumed to be 1. This accounts exactly for the loss given default (LGD), which makes the agreement very similar to an insurance contract (Bharath & Shumway, 2008). Nonetheless, the main difference is that the buyer of the CDS does not have to own the underlying financial instrument and consequently, these products encourage credit speculations.

In this subsection, T, the maturity of CDS, will equal five years unless stated differently. As the price of credit default swaps is often expressed in the CDS spread s, it is desirable to infer the implied yearly probability of default conditional to the fact that the default has not occurred earlier. In a similar manner as Bharath and Shumway (2008), I will denote the annualized implied PD as π^{CDS} and assume the default hazard rates to be constant, which is a standard simplification used for theoretical purposes (Tarashev & Zhu, 2006).

Nevertheless, a more complex approach known as the JP Morgan approach, which is often used in the financial industry, involves the construction of a piecewise constant default hazard curve. Therefore, it relaxes the assumption of constant default hazard rates. The JP Morgan approach is based on the CDS spread data for several maturities and hence, it requires more data (Castellacci, 2012). Indeed, such approach is also used by Bloomberg Professional (Wen & Kinsella, 2016). However, since the exact pricing of the CDS is not under the scope of this thesis, I use the simplified version instead.

Furthermore, for the sake of simplicity, Bharath and Shumway (2008) suppose that the potential default always happens in the middle of the given payment period. On the other hand, each buyer's payment occurs at the end of the payment period. As a result, within this model's framework the last accrual payment in case of default is made exactly in the middle of that payment period, amounting to $\frac{0.5s}{n}$ where $\frac{1}{n}$ stands for the payment frequency. Without the loss of generality the assumed notional of the CDS will be 1 USD.

The expected present value of the selling party's cash flows at the initiation of the CDS is reflected in the following equation (65). Here, r represents the continuously compounded risk-free rate, which is assumed to be constant in this simplified world (Bharath & Shumway, 2008). Additionally, $\pi^{CDS,non}$ stands for the non-annualized PD, which can be translated into the annualized PD as defined by the equation (66).

$$\sum_{i=1}^{nT}\left(\frac{(1-\pi^{CDS,non})^{i}e^{-rt_i}s}{n}+\frac{(1-\pi^{CDS,non})^{i-1}\pi^{CDS,non}e^{-r\left(t_i-\frac{0.5}{n}\right)}s}{2n}\right) \tag{65}$$

$$\pi^{CDS}=1-(1-\pi^{CDS,non})^{n} \tag{66}$$

The expression observed in the formula (65) sums up the discounted present values of expected cash flows in case of no default events in the first part. Moreover, an accrual payment in case of default is taken into account in the second term of the equation (Bharath & Shumway, 2008).

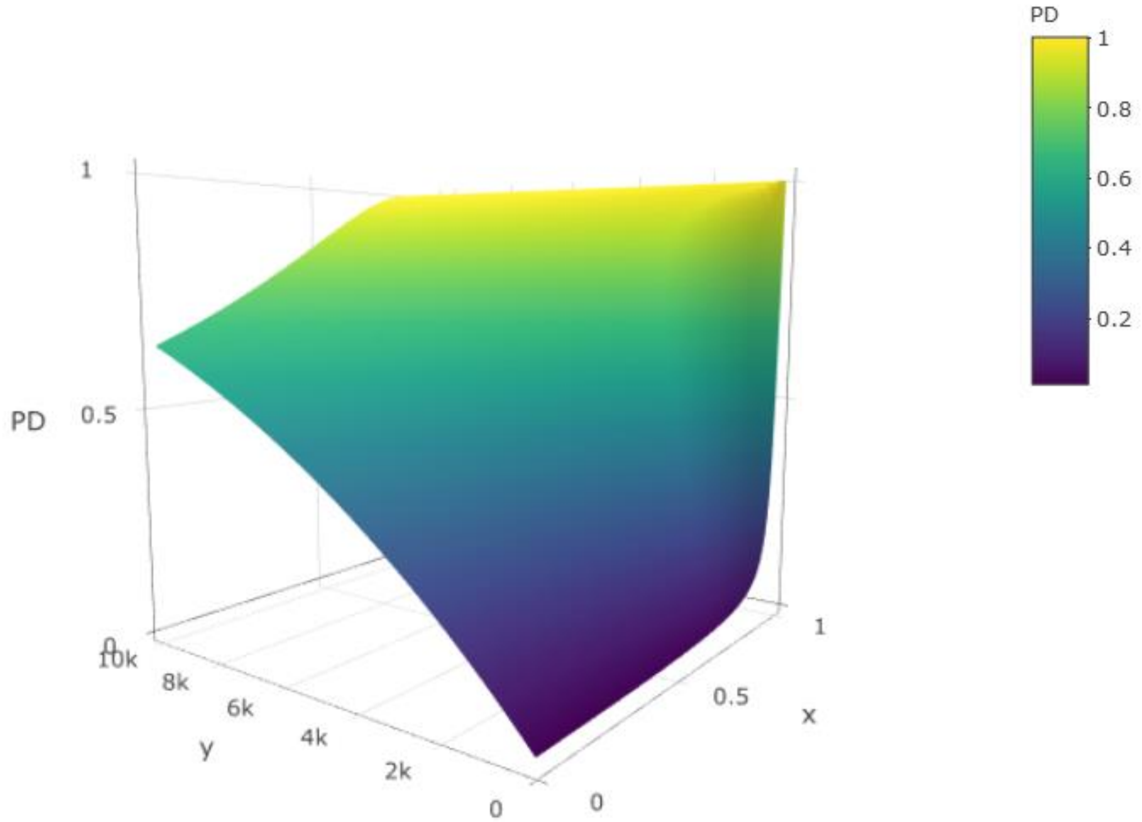
On the other hand, the expected present value of seller's cash flows equals the sum of all expected payments in case of issuer's default. This is given by the following formula (67) where δ is a constant recovery rate of the notional (Bharath & Shumway, 2008).

$$\sum_{t=1}^{nT}(1-\pi^{CDS,non})^{i-1}\pi^{CDS,non}(1-\delta)e^{-r\left(t_i-\frac{0.5}{n}\right)} \tag{67}$$

By the non-arbitrage argument, the expected present values of the future cash flows at the initiation are supposed to be in line. Therefore, the sums of cash flows obtained in the equations (65) and (67) should be equal, which produces the equation that implies a PD. However, in the aforementioned formulas one can detect two unknowns as neither δ nor $\pi^{CDS,non}$ are given, which makes a set of possible solutions for this equation where $\delta \in [0,1]$. Bharath and Shumway (2008) argued that the expected cash flows are almost independent of the recovery rate and thus, the modelling choice should not have a large impact on the valuation or PD estimation. By choosing some universal recovery rate, one ends up with one nonlinear equation where only one unknown parameter $\pi^{CDS,non}$ is left.

However, it has to be noted that in general a recovery rate has some impact on the implied annualized PD. Given the valuation model described above, I implied the PDs for a list of recovery rates from 0 to 100 % displayed in the y axis and CDS spreads from 0 to 10,000 basis points presented in the x axis on Figure 7. One can notice that especially for the CDS with higher spreads, the effect of changing the recovery rate is not entirely negligible. On the other hand, the impact in absolute terms is limited for those companies that are far from default, unless recovery rate is close to 1. This is in line with Beinstein and Scott (2006) where authors had similar observation. They claimed that it is intuitive since the recovery rate is more relevant for companies which are about to bankrupt or close to bankruptcy and a rather unimportant information for safe companies.

Figure 7: Implied annualized PD for a series of recovery rates (in %) and CDS spreads (in basis points). Payment frequency of this artificial CDS is quarterly, maturity of the instrument is 5 years and risk-free rate is assumed to be 2 %.



Source: own work.

Based on the literature, there exist several approaches for choosing the recovery rate. For instance, Bharath and Shumway (2008) choose a universal δ to be 25 %. Alternatively, Moody's recovery database does not simplify the problem but rightfully assigns rates based on the estimated percentage that investor can expect in case of liquidation of a firm's assets. According to Wen and Kinsella (2016), mean and median of Moody's recovery rates reach 37 % and 24 %, respectively. In a similar manner, the daily average of recovery rates in Markit's database is generally between 37 % and 40 % (Tarashev & Zhu, 2006).

On the other hand, Bloomberg (2017) model that computes implied PDs from CDS spreads simply assumes a fixed rate of 40 % for senior debt, regardless of the selected company and the economical background of the given valuation date. Since I do not possess Moody's firm specific recovery rate data, I follow Bloomberg's approach in the practical implementation of this model (see Appendix 6 for details).

Lastly, the discussed CDS pricing model above neglects the counterparty credit risk, which is a common assumption and in the spirit of Hull, Predescu and White (2004b) who claim that its effect is negligible.

2 DATA AND MODELLING APPROACHES

In this section, I present the input data, along with its corresponding sources, the calibrated variables and compare the summary statistics of the implied PDs. Besides, I briefly discuss what additional or alternative input data might be needed in case of enhanced studies and how it would be used. Additionally, I debate modelling choices used in order to implement the models.

2.1 Firm Selection Process

For the purposes of my master's thesis, I obtained most of the data from Bloomberg Professional (2017). In the sample of firms I chose those that have been a part of the S&P500 index as of February 1, 1990. Similarly to Bharath and Shumway (2008), I excluded all 56 financial companies which were listed in the S&P500 back then. The reason, why it may be preferred not to mix financial companies such as banks with other industries, is that financial institutions are special kind of companies with a different balance sheet structure. In addition, especially in the last decade financial firms have become a subject of increased regulations, which for example determine minimum the capital requirement that serves as a buffer and decreases the PD (Imerman, 2011).

Furthermore, I excluded all companies that have either been acquired or have been merged with some other company between February 1, 1990 and August 4, 2017. The main issue related to including these companies in the analysis is that they might have been close to default which was the reason for acquisition. This could produce biasness in the following analyses in section 3, when performing the regressions of time to default on variables such as estimated probabilities of default. In case any of these companies has been close to default, it would appear as if the default was not about to happen. Hence, it might produce corrupted results due to non-independent censorship. Henceforth, I skip 237 companies from the study, as they were either acquired or have merged during the analyzed window period.

In addition, the Bloomberg Professional (2017) database does not provide complete data for 14 other companies. More specifically, the data for at least one of the variables is missing in Bloomberg Professional (2017). Therefore, these companies have been removed from the analysis.

Finally, after the removal of each of these companies I end up with 164 companies that have not bankrupted and 29 companies that have defaulted or have been liquidated between February 1, 1990 and August 4, 2017 according to Bloomberg Professional (2017) database. I listed the analyzed firms in Appendix 10. All of these companies back then listed among S&P500 companies have sufficient data for the analysis and thus, I am able to assume that the data used is reliable.

2.2 Bankruptcy dates

According to the United States Bankruptcy Law, a US firm is considered to go into bankruptcy on a day when it files for Chapter 11. Starting on this date, a defaulted firm applies for a process, which could last for years. In this period it is supervised by a court and during this period of time the aim for the firm is to restructure its obligations. Finally, the financial problems of a company are supposed to get resolved. Empirical studies suggest that 95 % of the companies, which had filed for Chapter 11, escaped the liquidation. In case this is not feasible, a firm has to apply for Chapter 7 based on which the companies are eventually liquidated (Elizade, 2006).

Nevertheless, empirical researchers are not necessarily in agreement when it comes to a measure which determines bankrupted companies and bankruptcy dates. Campbell, Hilscher and Szilagyi (2008) describe two possible approaches for identification of firms that failed to meet their obligations. Firstly, they consider the date of bankruptcy as a date when a firm files for either Chapter 7 or Chapter 11. On the other hand, they claim that a broader definition of bankruptcy events could also consider financially driven delisted companies or companies that received D ratings by any of the main credit rating agencies (Campbell, Hilscher & Szilagyi, 2008).

On the other hand, Elizade (2008) has a different point of view as he supposes that the first passage models could only model the liquidation of companies. Therefore, in his paper a liquidation event corresponds with the model’s date of default and only companies that filed for Chapter 7 should be considered bankrupted when using the first passage models. When modelling the companies that filed for Chapter 11, the liquidation process models should be acquired as these companies are not necessarily going to be liquidated and indeed, they are generally not. As a result, they consider companies liquidated whenever the level of assets remains below the face level of debt for some period of time. Nonetheless, this type of models have not been empirically tested well enough yet (Elizade, 2008).

Within the scope of this thesis, I will consider the dates when firms filed for either Chapter 7 or Chapter 11 as the bankruptcy dates of these companies. This is consistent with the first approach discussed by Campbell, Hilscher and Szilagyi (2008).

2.3 Input Data

2.3.1 Structural Credit Risk Models Data

The balance sheet inputs for structural credit risk models consist of the current (Bloomberg data item: BS_CUR_LIAB) and the total liabilities (Bloomberg data item: BS_TOT_LIAB2) which produce F, the face value of debt, if one takes all current and half of the long-term liabilities. In addition to it, I acquire alternative data such as the net income available to common shareholders (Bloomberg data item: EARN_FOR_COMMON) and the book value of total assets (Bloomberg data item: BS_TOT_ASSET), which combine the net income to total assets ratio (NI/TA). Each mentioned balance sheet data is expressed in million USD in Bloomberg Professional (2017).

Next, the multiplication of the number of shares outstanding (Bloomberg data item: BS_SH_OUT), which are also expressed in millions, and the share price (Bloomberg data item: PX_LAST) results in E, the market value of equity. In a similar manner, I obtain a time series of S&P500 index values, which enables me to compute a firm's yearly excess return with respect to the benchmark index. I compute it as a simple difference between a firm's log return and S&P500's log return, and denoted it as $r_{it-1} - r_{mt-1}$.

Similarly to Bharath and Shumway (2008), and Vassalou and Xing (2004), I obtain a time series of 1-year Treasury rates which I generally assume to be r, the risk-free rate. Moreover, a spectrum of Treasury yield rates R_M with different maturities is sourced in order to calibrate the Vasicek parameters to the market data. The treasury yield rates are the only data that I acquired from the publicly accessible source – the Federal Reserve Bank of St. Louis (2017) web page – and not from the Bloomberg Professional (2017) database.

All aforementioned historical market data, including share prices, interest rates and S&P500 index, have generally been obtained on a daily basis. However, all balance sheet data such as short-term and total liabilities, total assets, shares outstanding and net income are in general refreshed quarterly.

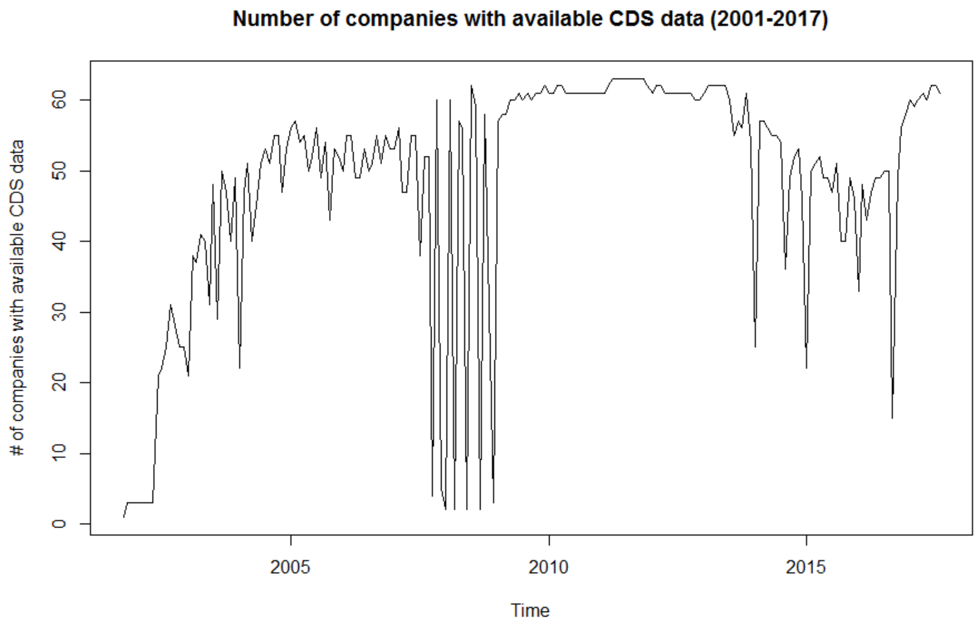
2.3.2 CDS Data

The products that carry market information about a firm's PD and are used in this thesis are credit default swaps. In particular, most of the liquid 5-year CDS with quarterly paid coupons have been chosen for the analysis. On the market, including the Bloomberg Professional (2017) database, one can usually observe the product price through the CDS spreads s, which are quoted in basis points (see Table 1). These were converted into the implied PDs as discussed in section 1.6.

Nonetheless, as these products exist only since 1994, there is less data available, especially up until June 2002 as displayed in Figure 8. Moreover, for our sample companies, Bloomberg contains data only since September 6, 2001 starting with IBM (International Business Machines Corp), Wal-Mart Stores Inc. and Altria Group Inc. CDS. Additionally, data points for many tenors are missing especially during the global financial crisis period 2007–2008. As a result, all Cox proportional hazard regressions performed in section 3, where the CDS implied PD is one of independent variables, have been affected in a sense of a lower number of observations in this period. Nonetheless, I do not replace missing data with the last observation of the CDS spread in order not to corrupt the data and therefore influence the results of the analyses.

All in all, I got data for 63 firms whereas data for 130 has not been extracted from Bloomberg Professional (2017). Indeed, Bharath and Shumway (2008) deal with a similar issue as balance sheet data has been sourced for more than 15 thousand firms, whereas the CDS spread data *s* covered approximately 4 times fewer firms and three times smaller time window, which may have slightly compromised their final results.

Figure 8: Number of analyzed companies with available CDS spread data in Bloomberg Professional (2017) between 2001 and 2017.



Source: own work.

2.3.3 Summary Statistics

In the table below I exhibit means, standard deviations and quantiles of all input data needed for the implementation of structural credit risk models and for the implication of PD from the CDS data. Apart from that, I also display summary statistics for variable NI/TA used in order to challenge structural credit risk models' importance in section 3.3.

Table 1: Summary statistics of the input data

Variable notation	N	Mean	Std. Dev.	Quantiles				
				Min	25 %	Median	75 %	Max
F	1,216,107	11.68	30.52	0.02	1.36	4.11	10.31	490.40
E_t	1,211,563	24.26	51.19	0.00	2.42	7.57	21.33	812.63
NI/TA	58,553	0.01	0.03	–1.23	0.01	0.01	0.02	0.78
r	6,874	3.11	2.38	0.08	0.52	3.35	5.29	8.64
s	9,206	121.37	243.98	5.25	33.11	56.82	114.72	7,239.36
$r_{it-1} - r_{mt-1}$	55,810	-1.22	33.97	-419.11	-16.33	-0.30	15.46	452.26
R_M	66,825	3.90	2.30	0.00	1.92	4.24	5.67	9.18

This table reports summary statistics, including quantiles, means, standard deviations and number of observations for each input variable. One should note that quantiles, means and standard deviations for F and E_t are expressed in USD billions, CDS spread s is denoted in basis points, whereas Treasury yields R_M, risk-free rate estimate r and excess returns $r_{it-1} - r_{mt-1}$ are standardly presented in percentages. Apart from that, the number of observations N is always displayed as an integer.

Source: own work.

In Bharath and Shumway (2008) it can be observed that the means and all quantiles of balance sheet items, such as the market value of equity and the face value of debt, are approximately 30 and 50 times lower, respectively. This can be explained through the following two reasons. Firstly, the values of the items in balance sheets in 1990–2017 increased significantly in comparison to 1980–2003 used by Bharath and Shumway (2008). Secondly, it has to be noted that Bharath and Shumway (2008) used a wider range of companies. More specifically, they accounted for all NYSE, AMEX and NASDAQ companies, whereas my analysis is based on the S&P500 companies only. Therefore, the balance sheet items in my sample of companies are in average substantially higher as these are generally considered to be the largest ones in the market.

On the other hand, the average 1-year Treasury yield decreased dramatically, which is especially due to the post-crisis period that my dataset accounts for. The average CDS spread is lower by 44.5 bps in my sample; however, the standard deviation is higher which results in more extreme outliers as well. In particular, this is a consequence of the financial crisis when CDS spreads generally skyrocketed.

2.4 Variables Implied or Calibrated from the Input Data

2.4.1 Equity Return Volatility

One additional set of data needed is σ_E, which is essential when calibrating a market asset return volatility. I decided to compute it historically as a standard deviation of past year's equity market values E. Alternatively, one could calculate it from the option implied volatility data, however, some firms may not have corresponding options or they might be less liquid and produce many missing data. A historical volatility is a standard procedure for theoretical purposes, however, the alternative approach, which involves implied volatility data, could in general produce slightly better predictive performance as observed by Bharath and Shumway (2008).

Let me mention that the option-implied volatility surface data in the market consists of maturity and moneyness dimensions. Whereas the maturity in our case is clearly set to 1 year, the moneyness should generally be considered too. However, these modelling choices are not as straightforward as it appears. For example, Bharath and Shumway (2008) opted for the 30-day maturity ATM (at-the-money) call option implied volatility data, however, there is no clear reason for choosing such maturity. On the other hand, the moneyness component is generally taken into account in financial industry for option pricing purposes too. Indeed, it may be an over-simplification to take the ATM data, especially since volatility can drastically affect the option price. However, one should be aware that in the structural credit risk models the idea of an option is purely artificial and thus, the volatility of assets, which is one of the main inputs, is not observable in the market.

Nonetheless, Hull, Nelken and White (2004a) examined how neglecting the equity volatility skew in the traditional Merton model affects its efficiency. They viewed the market equity option data as artificial options on the call European option with underlying being the firm value. That said, the equity options were indeed considered as compound options. By using such approach, they implied asset volatility directly by using the option pricing theory rather than relying on optimization techniques used in the Merton DD implementation presented in section 1.1.2. Their results suggest that by taking the volatility skew into account, the model outperforms the traditional Merton model. Overall, in order to simplify the data import process and not to exceed the frames of the thesis, I do not consider the implied volatility data.

When dealing with time series in practice, i.e. the estimation of yearly returns or the estimation of volatility based on historical data, it is common to follow the day count conventions. I follow the BUS/252 day count convention and count the fraction of the year between the two dates as already specified in the equation (35). Therefore, for the estimation purposes of historical volatility I calculate the equity volatility over the last 252 data points.

2.4.2 Correlation

The LS and CDG models also require the estimation of the correlation factor between the two standard Brownian processes that generate the randomness of the two models. In the equation (29) I denoted it as ρ and it represents the correlation between the risk-free rate and the return on the market asset value. As an approximation, Eom, Helwege and Huang (2004) estimated the correlation of equity returns with respect to the 3-month T-bills.

Since I only have the market asset value data on monthly basis, I follow the same approach and calculate the correlation between the equity returns and 1-year Treasury yield, which I assume to be a proxy for the risk-free rate. As I have the daily data for these two time series for each company at my disposal, I estimate ρ based on historical daily observations in a 10-year window. In order not to lose too many observations for the first years of the data set, namely between 1991 and 2000, I also take into account the computed correlation on the existing history for dates with at least 1 year of data. Both 1 year and 10 years of data are defined with the day counting convention BUS/252 that has already been presented in the equation (35).

Nonetheless, the decision of picking a time window for the calculation of the correlation is arbitrary and it generally depends on the modelling choice of modelers. For example, Eom, Helwege and Huang (2004) compute the correlation based on a 5-year time window, whereas Suo and Wang (2006) take all existing history into account.

2.4.3 Summary Statistics

Apart from the equity volatility σ_E and correlation ρ, one also has to calibrate the Merton DD variables σ_V, V_t and μ_V, and the Vasicek variables β, θ and η in order to implement the structural credit risk models presented in section 1. Apart from that, the computation of Naïve asset return $\mu_{V,Naive}$ and volatility $\sigma_{V,Naive}$ is needed for benchmarking the Naïve model against the Merton DD model. In the following table I exhibit all corresponding summary statistics. All values are presented in percentages, except of market value of assets V_t and the Vasicek parameter β which are expressed as USD billions and integers, respectively.

Table 2: Summary statistics of implied or calibrated variables

Variable notation	N	Mean	Std. Dev.	Quantiles				
				Min	25 %	Median	75 %	Max
σ_E (%)	1,163,901	33.62	19.30	5.99	21.71	28.74	39.34	394.26
ρ (%)	58,790	–0.43	2.82	–20.02	–1.83	–0.52	0.91	23.56
σ_V (%)	55,688	22.49	13.11	3.63	14.59	19.54	26.66	319.95
V_t (bn USD)	55,688	36.89	73.96	0.05	4.49	13.17	33.60	932.74
μ_V (%)	53,355	5.85	23.22	-233.41	-5.54	6.57	17.93	173.39
β	331	0.35	0.55	0.03	0.14	0.21	0.33	5.00
θ (%)	331	6.83	2.06	2.72	5.18	6.49	8.69	10.00
η (%)	331	3.10	2.60	0.10	0.87	2.51	4.60	14.53
$\mu_{V,Naive}$ (%)	55,994	6.01	36.94	-481.39	-9.59	8.65	24.71	450.90
$\sigma_{V,Naive}$ (%)	55,712	25.31	11.80	9.07	17.61	22.52	29.77	231.81

This table reports summary statistics, including quantiles, means, standard deviations and number of observations, for each implied or calibrated variable. One should note that quantiles, means and standard deviations for V_t are expressed in USD billions, Vasicek's parameter β is denoted in integers, whereas other variables are presented in percentages. Apart from that, the number of observations N is always displayed as an integer.

Source: own work.

Firstly, when benchmarking my calibrated Merton DD variables against Bharath and Shumway (2008), it can be observed that my market asset values are on average 34 times higher. This is consistent with the higher values of balance sheet items in my sample of companies as noted in section 2.3.3.

Secondly, Suo and Wang (2006) reported the means and standard deviations of calibrated Vasicek parameters between 1983 and 2002. The average expected long-term average θ is well in line with my results, whereas their mean reversion speed β and short-rate volatility η are approximately 4 times lower in their study. The direct comparison cannot be assessed though as the analyzed time window is not identical.

Finally, the naïve estimates $\mu_{V,Naive}$ and $\sigma_{V,Naive}$ are relatively closely in line with the corresponding Merton DD predictors in Table 2. Bharath and Shumway (2008) came up with a similar conclusion. This observation makes the question, whether the calibration used to build the Merton DD model adds to the predictive power of the model, even more relevant. I inspect it in subsection 3.4.

2.5 Probability of Default Results

2.5.1 Structural Credit Risk Model Predictors

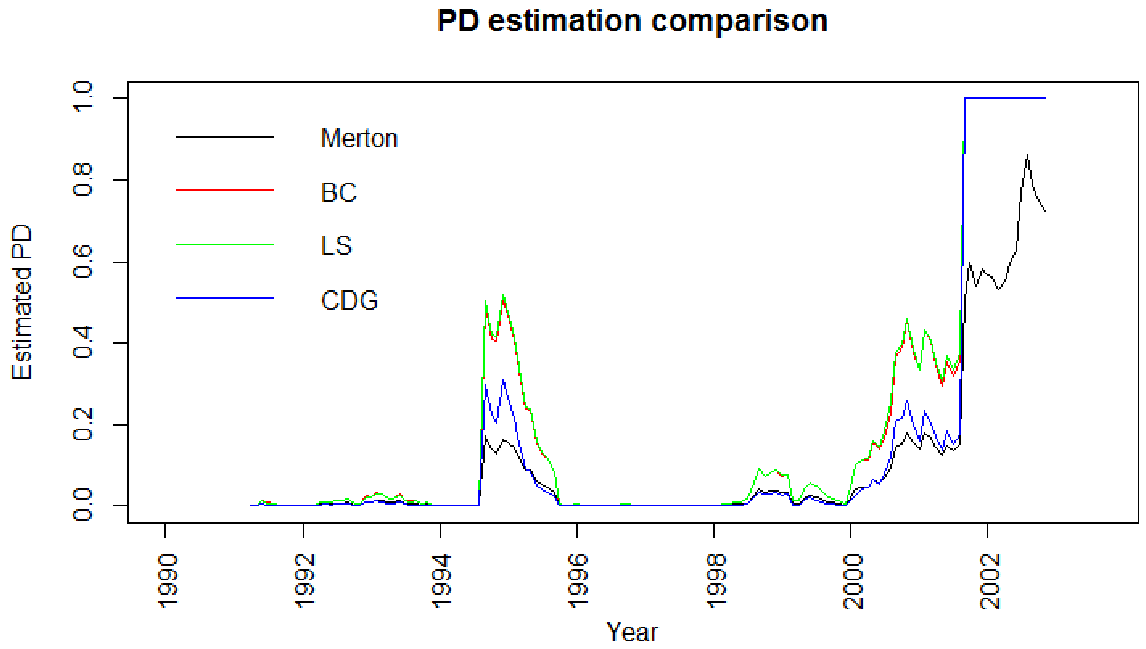
Bharath and Shumway (2008) produced an alternative PD estimator $\pi_{\mu=r}^{Merton}$ in a similar fashion as π^{Merton} is calibrated in section 1.1.2. The only difference between the two approaches is a replacement of the estimated annualized returns μ_V for a risk-free interest rate r in formula (18), which is considered in the initial theoretical Merton model. In other words, $\pi_{\mu=r}^{Merton}$ is the Merton PD estimator under the risk-neutral measure, whereas the Merton DD model produces the physical PD estimates. As a result, when benchmarking the results between the two Merton estimates I capture the benefit, if any, of the annualized asset returns estimation in the Merton DD model.

Additionally, it is sensible to evaluate the impact of the setting the barrier parameter γ to r in the equation (26) in the implementation of the Black-Cox model PD predictor π^{BC} as many papers, such as Elizade (2006), suggest. However, for the purposes of the direct comparison with respect to the LS model I consider special case where γ is set to 0. Thus, $\pi_{Cons\ T}^{BC}$ estimates a PD based on the assumption that the threshold is constant which is equivalent to the LS model boundary.

In practice, many practitioners, for instance Huang and Zhou (2008) or Brigo and Tarenghi (2005), calibrate structural credit risk models to CDS data and present the Merton model's theoretical extensions in details. Additionally, the papers also discuss the calibration of asset market value V_t, its volatility σ_V, or the face value of debt F within the Merton model framework. However, at least according to my knowledge, there are no papers that would clearly discuss how to calibrate these parameters within the BC, LS and CDG model frameworks.

Therefore, I apply the Merton DD calibration algorithm which is described in section 1.1.2, in order to determine the market value of assets for the BC, LS and CDG models as well. In addition to it, I follow the modelling decision of choosing the face value of debt F as Merton's practical implementations in the equation (15). Consequently, the comparison between the PD estimators $\pi_{\mu=r}^{Merton}$, $\pi_{Cons\ T}^{BC}$, π^{LS} and π^{CDG}, namely the risk-neutral Merton, the BC, LS and CDG estimators, respectively, does not depend on the calibration approach of the parameters V_t and σ_V. On the contrary, the performance of structural credit risk models in section 3 rather depends on their assumptions and their functional forms which is the main purpose of this thesis. The graphical comparison of the four structural credit risk models is illustrated in Figure 9. In particular, the PD estimations are displayed for exemplary company UAL (United Airlines) Corp. which filed for Chapter 11 on December 9, 2002. It is noteworthy that all but Merton model indicate $PD = 1$ in the last year before the company filed for Chapter 7, which suggests that Merton model may underestimate the PD with respect to the other structural credit risk models.

Figure 9: Graphical comparison of PD estimation dynamics among structural credit risk models performed on UAL Corp.



Source: own work.

2.5.2 Summary Statistics and Correlation Estimates

According to the theory presented in section 1, $\pi_{\mu=r}^{Merton}$ should be lower than both π^{BC} and $\pi_{Cons\,T}^{BC}$ for each observation and each firm. The rationale is that the Black-Cox model introduces additional barrier, which results in a company's default if this threshold is breached. Additionally, the difference between $\pi_{Cons\,T}^{BC}$ and π^{LS} accounts for the stochastic interest rate contribution, whereas the comparison between π^{LS} and π^{CDG} suggests the added value of non-deterministic obligations.

The structural credit risk model estimates $\pi_{\mu=r}^{Merton}$, π^{BC}, $\pi_{Cons\,T}^{BC}, \pi^{LS}$ and π^{CDG} cover approximately 55,688 points, π^{Naive} produces 7 estimates more, whereas π^{Merton} consists of 53,355 data points. On the other hand, note that as opposed to the other variables, the CDS data covers only the data from 2001 on. Hence, the CDS data contains 9,040 data points only. In the top panel of Table 3 I examine all PD estimates produced by the structural credit risk models and the PDs implied from the CDS. In the panel B I display the Pearson correlations between the PD estimates.

Table 3: Summary statistics of PD estimates and correlation matrix

Panel A: Summary Statistics

Variable notation	N	Mean	Std. Dev.	Quantiles: Min	25 %	Median	75 %	Max
π^{Merton} (%)	53,355	4.01	14.57	0.00	0.00	0.00	0.04	100.00
$\pi^{Merton}_{\mu=r}$ (%)	55,688	2.20	8.75	0.00	0.00	0.00	0.03	99.90
π^{Naive} (%)	55,695	6.36	19.98	0.00	0.00	0.00	0.14	100.00
π^{BC} (%)	55,688	3.78	14.00	0.00	0.00	0.00	0.07	100.00
$\pi^{BC}_{Cons\,T}$ (%)	55,688	4.00	14.69	0.00	0.00	0.00	0.07	100.00
π^{LS} (%)	55,688	4.00	14.68	0.00	0.00	0.00	0.07	100.00
π^{CDG} (%)	55,688	3.19	13.38	0.00	0.00	0.00	0.02	100.00
π^{CDS} (%)	9,206	1.93	3.37	0.09	0.55	0.94	1.89	70.33

Panel B: Correlation Matrix

	π^{Merton}	$\pi^{Merton}_{\mu=r}$	π^{Naive}	π^{BC}	$\pi^{BC}_{Cons\,T}$	π^{LS}	π^{CDG}	π^{CDS}
π^{Merton}								
$\pi^{Merton}_{\mu=r}$	0.7881							
π^{Naive}	0.8464	0.6085						
π^{BC}	0.8043	0.9837	0.6396					
$\pi^{BC}_{Cons\,T}$	0.8049	0.9743	0.6468	0.9962				
π^{LS}	0.8052	0.9740	0.6479	0.9958	0.9999			
π^{CDG}	0.7679	0.9522	0.5893	0.9589	0.9582	0.9581		
π^{CDS}	0.6262	0.6790	0.6028	0.6850	0.6813	0.6822	0.5773	

Panel A reports summary statistics, including quantiles, means, standard deviations and number of observations for each PD estimate. One should note that quantiles, means and standard deviations for each structural credit risk model estimate are presented in percentages. Apart from that, number of observations N is always displayed as integer. On the other hand, Panel B displays Pearson correlation matrix among the PD estimates. The correlations are expressed as integers.

Source: own work.

The table above reflects the aforementioned theory as the means and quantiles of both BC estimates are above the corresponding summary statistics of $\pi^{Merton}_{\mu=r}$. Hence, it can be claimed that the BC model always produces more conservative estimates than the Merton model. On the other hand, the Naïve model produces higher average estimates than the analyzed structural credit risk models, and they are only slightly below the average Naïve estimates reported in Bharath and Shumway (2008). Furthermore, the, average π^{Merton} is almost perfectly in line with Vassalou and Xing (2004), while Bharath and Shumway (2008) on average observed more than twice higher predictions.

The average CDS implied PD is slightly below 2 % which is only 20 basis points lower than the average CDS implied PD calculated by Bharath and Shumway (2008). Therefore, one can claim that the above results are reasonable and generally fairly in line with similar studies.

Finally, a meaningful observation is that the BC model with a constant barrier and the LS model produce almost identical results, which is reflected both in Figure 9 and Table 3. In addition, Panel B suggests that the BC and LS models produce estimates that are almost perfectly correlated. This brings up a question, whether the calibration of Vasicek parameters and the implementation of computationally more consuming LS PD algorithm contribute to the PD estimation. Based on my data sample I aim to answer this question in section 3.

3 EMPIRICAL RESULTS

In this section I display and discuss the outcomes of empirical analyses, such as the comparison of PD sensitivities between structural credit risk models, and the results of Cox proportional hazard regressions. The latter method is needed to test the importance and sufficiency of structural credit risk models that has been described in section 1.5. Additionally, the comparison among the models through backtesting will be presented.

3.1 Sensitivity Analysis

In this subsection I will be interested in the sensitivities of the four analyzed structural credit risk models. Greeks in derivatives pricing world and sensitivities are generally often calculated numerically in practice. More specifically, one-dimensional sensitivities are calculated by bump-and-revalue principle, meaning that one bumps the observed parameter and keeps all other variables untouched. The sensitivity of the observed parameter is finally calculated as a difference between the price, or the PD in our case, after the bump and the original value. In Table 4 I bump each parameter of the model in absolute terms based on the bump value specified in column 3. For example, Merton's PD is originally 9.1165 %, however if we bump the value of assets by 1bn USD, the PD decreases to 8.9737 %. Therefore, the sensitivity of $\pi_{\mu=r}^{Merton}$ with respect to V_t equals -14.27 basis points.

In Table 4 I present all variables in column 1, their original values in column 2 and the sizes of the corresponding bumps in column 3. All sensitivities are exhibited in columns 4–7, depending on the model, and expressed in basis points. In particular, the analysis is conducted on General Electric (GE) as of August 3, 2009.

Table 4: Comparison of parameter sensitivity among structural credit risk models.

Variables	Base values	Bump value	Structural credit risk model's sensitivity			
			$\pi_{\mu=r}^{Merton}$	$\pi_{Cons\,T}^{BC}$	π^{LS}	π^{CDG}
PD (%)	-	-	9.12	17.64	17.49	13.58
V_t	581.62bn	1bn	-14.27	-27.45	-27.21	-22.28
F	441.31bn	1bn	19.06	36.66	36.34	29.82
σ_V	19.62 %	1 %	128.19	240.97	237.96	231.30
r	0.48 %	1 %	-80.76	-119.57	-111.28	-141.17
ρ	2.12 %	1 %	-	-	3.08	-0.56
β	14.80 %	1 %	-	-	-4.65	-5.74
θ	10.00 %	1 %	-	-	-7.01	43.61
η	4.77%	1 %	-	-	22.04	21.70
λ	0.18	1 %	-	-	-	-16.64
ν	0.6	1 %	-	-	-	-18.55
ϕ	2.8	1 %	-	-	-	1.77

In this table I report one-dimensional sensitivities calculated by bump-and-revalue principle for General Electric as of August 3, 2009. The original PD for each structural credit risk model is displayed in the top row and expressed in percentages. On the other hand, column 2 presents values of each variable, column 3 reports size of a bump, whereas the last 4 columns present calculated sensitivities in basis points. Note that for Merton, BC and LS models PDs are not sensitive to some parameters used only by the more complex models. This is why some sensitivities are not reported (-).

Source: own work.

From the table above it can be noticed that all models suggest reasonable sensitivities with respect to the base parameters, namely market asset value, face value of debt, asset return volatility and risk-free rate. Indeed, the increase of market asset value leads towards a decreased PD and a higher face value of debt has to generate larger default probabilities. Moreover, it is intuitive that high volatility results in a higher probability of default and that it presents one of the most important parameters. What is more, an increased risk-free rate suggests a higher growth and consequently, a lower PD. This is in line with Collin-Dufresne and Goldstein (2001).

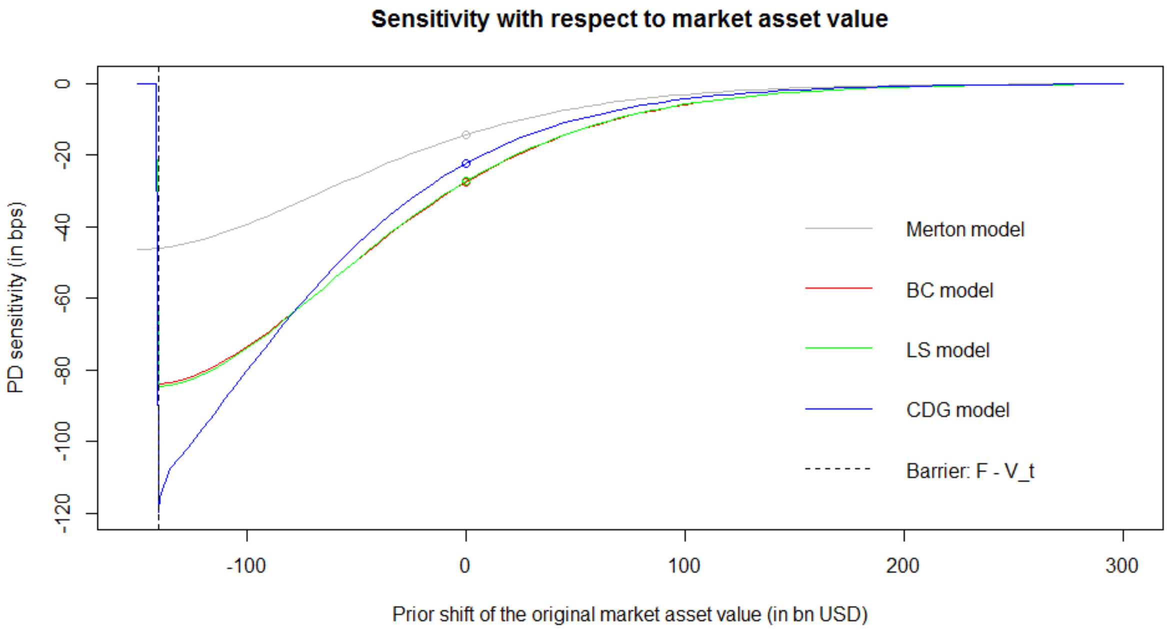
The magnitudes of PD sensitivities in absolute terms are in general the lowest for the Merton model. This is logical as, for example, a further rise of market asset value would be more valuable for the BC model in order to be further away from the barrier. The idea is more generally illustrated in Figure 10.

In Figure 10 I display the dynamics of PD sensitivities with respect to the 1bn USD market asset value bump. For every point on the graph, the x axis defines the prior shift of the original market value of assets. Hence, PD sensitivities, where the x coordinate equals 0 in Figure 10, are in line with those exhibited in Table 4 in row devoted to V_t. PD sensitivities with the prior shift equaling 0 are pointed out as circles in Figure 10.

In order to shed some additional light on the reasoning of the graph, I explain the idea using an example. When the x coordinate of a data point is -140 bn USD, V_t are prior decreased from 581.62bn USD to 441.62bn USD. Thus, V_t is just above the barrier based on the column *Base values* in Table 4. Consequently, the additional 1 bn USD bump of market value of assets would result in a significant decrease of PD, which is why the PD sensitivity is negative for all first passage model types. In this case, same applies to the Merton model PD sensitivity, even though the barrier is by construction not considered in this framework.

In Figure 10, it can be seen that the BC market asset value sensitivity is above Merton's whenever the market value of assets is above the barrier. This is due to the fact that for the BC, LS and CDG models the difference between the face value of debt and V_t presents a borderline as a PD beyond this limit is fixed at 1. Hence, a minor move away from the barrier, presented as a dashed black vertical line in Figure 10, is much more meaningful than within Merton's framework. Following a similar idea, an increased debt would hurt first passage model types more severely whenever debt is above the assets level.

Figure 10: Comparison of market asset value sensitivities among structural credit risk models. Displayed as of August 3, 2009 for General Electric.



Source: own work.

On the other hand, the sensitivities of Vasicek's parameters have to be interpreted even more carefully. Given the parameters in the above example, it can be seen that the Vasicek long-term risk-free rate average θ is substantially higher than r corrected for the factor $\frac{\eta^2}{\beta}B(T-t,\beta)$. As a result, an increase in Vasicek's mean-reversion intensity factor β contributes to the rise of the risk-free rate which triggers an increase in the asset value. Therefore, it is reasonable that bumping β reduces a PD. However, the sensitivity of risk-free volatility η is in line with expectations as the parameter increases risk for both the LS and CDG models.

On the contrary, the former and the latter models are not in line when it comes to sensitivity with respect to the risk-free rate long-term average θ. When it comes to the LS model, θ only affects a PD through a positive effect on a risk-free rate which increases the asset value and decreases the PD. However, in the CDG model one also has to consider the model's assumption that a rise in θ contributes to the rise of debt and thus, a PD as presented in the equation (44). In our example, the latter overrules the risk-free rate effect as λ is large enough.

Similarly, the LS and CDG models are not in agreement for the sensitivity of the correlation factor between the two Wiener processes that build the two models. As expressed in the formula (71), the correlation factor influences assets in a negative way which results in an increase of a PD. However, Collin-Dufresne and Goldstein (2001) point out that the effect of correlation is not straight-forward and could go either way depending on other parameters. In particular, higher β changes the CDG PD sign from positive to negative.

The CDG specific barrier parameters may be even less intuitive to interpret. The λ effect and its size largely depend on the value in the brackets of the equation (44). As this value is negative, it negatively impacts debt which reduces a PD. However, an increase in ν, the target log leverage ratio, could cause a change in the direction of the influence. On the other hand, ν sensitivity is negative since the higher target log leverage ratio is, the higher negative effect it will have on the debt. Consequently, the PD will deteriorate. Finally, the leverage ratio sensitivity with respect to the risk-free interest rate, ϕ, positively effects the PD as θ is higher than the current risk-free rate so a potential for a rise of rates is high. This implies debt rise and thus, it amplifies the PD.

Overall, it can be claimed that for the main parameters the sensitivities among different structural credit risk models do not differ in a way that one could argue supremacy of one of the analyzed structural credit risk models. However, the LS and CDG models possess more parameters where signs of sensitivities are not necessarily unambiguous and hence, they deteriorate the simplicity of Merton and BC models.

3.2 Relevance of PD Predictors in Univariate CPH Regressions

In this subsection I display empirical evidence that all structural credit risk models, including the Naïve model, have predictive power in explaining the time to default of analyzed companies. I assess it through the Cox proportional hazard regression which I theoretically presented in section 1.5 and practically in Appendix 9. In each of the below analyzed regression models there is only one independent variable – a structural credit risk model PD estimate – while the time to default represents a dependent variable. In particular, I will test whether PD estimates are significant predictors, which can be assessed through p-values produced by the t-test.

Table 5 below reports results produced by the above discussed regressions. In particular, the estimates and corresponding robust standard errors, which are presented under the estimate in parenthesis, are exhibited. In line with standard notations, I denote estimates that are significant for 1 %, 5 % and 10 % significance levels as *, ** and ***, respectively. Besides, the results of LR test and the number of observations of each CPH regression are reported.

Table 5: Comparison of univariate Cox proportional hazard regressions of time to default on structural credit risk model estimates for period 1991–2017.

Dependent Variable: Days to Default

Regression equation: $\log h_i(t) = \alpha(t) + x_i(t)'\beta$

Independent Variables	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7
π^{Merton}	2.704*** (0.241)						
$\pi^{Merton}_{\mu=r}$		3.736*** (0.560)					
π^{Naive}			2.299*** (0.191)				
π^{BC}				2.606*** (0.331)			
$\pi^{BC}_{Cons\,T}$					2.540*** (0.324)		
π^{LS}						2.543*** (0.325)	
π^{CDG}							2.389*** (0.335)
N	53,355	55,688	55,695	55,688	55,688	55,688	55,688

Continues

Continued

LR test	1,770 (0.000)	1,560 (0.000)	2,250 (0.000)	1,734 (0.000)	1,818 (0.000)	1,823 (0.000)	1,347 (0.000)

For each univariate CPH regression, this table displays the number of observations, estimates and robust standard errors (discussed in section 1.5.1), which can be observed in parenthesis. I employed the default estimation method in R, namely Efron approximation. In addition, in the last row of the table I also exhibit the results of likelihood ratio (LR) tests, in particular test statistics and p-values (in parenthesis). Since all p-values are observed to be 0, the LR tests reject the null models and accepts the above specified models. In line with standard notations, I denote estimates that are significant for 1 %, 5 % and 10 % significance levels as *, ** and ***, respectively.

Source: own work.

As a consequence of the above results, one can claim that all 7 models are relevant and have some predictive power. More specifically, the outcome of, for example, Model 1 can be interpreted in a similar fashion as the log-level regression model. By holding the rest constant, a 1 percentage point bump of the Merton PD increases the daily hazard rate of default by $e^{2.704\,\%} - 1 = 2.74$ percent, which is implied from the definition in (64). A positive impact is indeed expected as an increased estimation of the PD should increase riskiness and thus, the hazard rate of default. Same interpretation can be applied to the other 6 models and other regressions presented in subsection 3.3. In addition, I re-perform the same analysis on the sub period 2001–2009 only. When performing the analysis on this sub period, I supposed that all information observed after December 1, 2009 is unknown. The reason for choosing this sub period is that majority of defaults occurred in this time period due to the global financial crisis. Therefore, the results of analysis for this sub period are supposed to be the most reliable. However, only one default has been detected since 2011 and thus, the sub period consisting of recent years is not considered. Regardless, similarly as in Table 5, the null hypotheses that $\beta = 0$ are rejected even at 1 % significant level. Therefore, independently of the chosen period, all 7 models have some predictive power in the univariate CPH regressions.

Table 6: Comparison of univariate Cox proportional hazard regressions of time to default on structural credit risk model estimates for sub period 2001–2009.

Dependent Variable: Days to Default

Regression equation: $\log h_i(t) = \alpha(t) + x_i(t)'\beta$

Independent Variables	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7
π^{Merton}	2.978*** (0.312)						
$\pi^{Merton}_{\mu=r}$		4.397*** (0.589)					

Continues

Continued

π^{Naive}			2.684*** (0.314)				
π^{BC}				2.865*** (0.412)			
$\pi^{BC}_{Cons\,T}$					2.838*** (0.412)		
π^{LS}						2.844*** (0.414)	
π^{CDG}							2.752*** (0.385)
N	19,036	19,067	19,052	19,067	19,067	19,067	19,067
LR test	841 (0.000)	781 (0.000)	1,087 (0.000)	799 (0.000)	846 (0.000)	851 (0.000)	722 (0.000)

For each univariate CPH regression, this table displays the number of observations, estimates and robust standard errors (discussed in section 1.5.1), which can be observed in parenthesis. Only sub period 2001–2009 has been accounted for in these regressions. I employed the default estimation method in R, namely Efron approximation. In addition, in the last row of the table I also exhibit the results of likelihood ratio (LR) tests, in particular test statistics and p-values (in parenthesis). Since all p-values are observed to be 0, the LR tests reject the null models and accepts the above specified models. In line with standard notations, I denote estimates that are significant for 1 %, 5 % and 10 % significance levels as *, ** and ***, respectively.

Source: own work.

Overall, the results observed in this subsection are similar to conclusions of Bharath and Shumway (2008) who tested and confirmed the relevance of the π^{Merton}, $\pi^{Merton}_{\mu=r}$, and π^{Naive}. The conclusion related to the relevance of structural credit risk model in univariate regressions is a necessary, but not a sufficient, condition to assure the general structural credit risk models' sufficiency and importance. I attempt to evaluate the two in section 3.3.

3.3 Importance and Significance of PD Predictors in Multivariate CPH Regressions

In this section I examine the multivariate regressions and try to assess whether any of structural credit risk models is a sufficient or at least an important predictor. Sufficiency is not expected as several alternative predictors, including CDS data, are supposed to provide additional information and therefore, are likely to be significant variables in regressions below. Indeed, this is what Bharath and Shumway (2008) observed in their analysis which covered the Merton model and Naïve model only. I examine the Cox proportional hazard regressions, combining one structural credit risk model variable and alternative independent variables in Table 7, Table 8 and Table 9.

In order to prevent the issues related to multicollinearity, I made sure that non-univariate Cox proportional hazard regressions pass the VIF (variance inflation factor) test of multicollinearity. The VIF test for the Cox proportional hazard models in R is implemented as function *vif* in package *rms* (Harell, 2018). Even though the threshold for such test is arbitrary, it is generally considered that VIF values below 10 for each independent variable suggest that the test did not detect multicollinearity. For the models presented in Table 7, Table 8 and Table 9, the highest VIF value is 6.27 for $\pi_{\mu=r}^{Merton}$ estimate in Model 2 in Table 9 but less than 4 elsewhere. Hence, apart from Model 2 in Table 9, which has to be interpreted with caution, I assume, that the results in the tables below, are generally reliable and valid for interpretation.

The models in Table 7 include variables that are directly or indirectly accounted in the structural credit risk model data but also other alternative variables such as NI/TA. Even though initially desired, the excess annual return had to be dropped from the analysis in order to prevent multicollinearity issues. On the other hand, I re-did the same analysis on sub period 2001–2009 in Table 8, whereas Table 9 accounts for models which incorporate the π^{CDS} estimate. Similarly as before, variables such as $ln(F)$ and $r_{it-1} - r_{mt-1}$ had to be dropped from the analysis in order to avoid multicollinearity issues as per the VIF criteria. In addition, it can be noted that sample sizes in regressions in Table 9 are substantially lower than in those presented in Table 7. This is expected as less CDS spread data has been obtained for sub period 2001–2017.

Table 7: Comparison of Cox proportional hazard regressions of time to default on structural credit risk model estimates and other explanatory variables for period 1990–2017.

Dependent Variable: Days to Default							
Regression equation: $\log h_i(t) = \alpha(t) + x_i(t)'\beta$							
Independent Variables	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7
π^{Merton}	1.261*** (0.276)						
$\pi_{\mu=r}^{Merton}$		1.702*** (0.637)					
π^{Naive}			1.249*** (0.201)				
π^{BC}				1.235*** (0.401)			
$\pi_{Cons\,T}^{BC}$					1.279*** (0.396)		
π^{LS}						1.285*** (0.397)	

Continues

Continued

π^{CDG}							0.948** (0.415)
$ln(F)$	-0.152 (0.125)	-0.141 (0.121)	-0.148 (0.122)	-0.145 (0.119)	-0.147 (0.118)	-0.147 (0.118)	-0.137 (0.120)
$1/\sigma_E$	-0.501*** (0.090)	-0.518*** (0.092)	-0.480*** (0.087)	-0.503*** (0.091)	-0.493*** (0.092)	-0.493*** (0.092)	-0.540*** (0.096)
NI/TA	-3.742*** (0.524)	-4.145*** (0.488)	-3.586*** (0.513)	-3.986*** (0.495)	-3.935*** (0.496)	-3.931*** (0.496)	-4.190*** (0.490)
N	53,355	55,688	55,695	55,688	55,688	55,688	55,688
LR test	3,734 (0.000)	3,890 (0.000)	4,161 (0.000)	3,936 (0.000)	3,993 (0.000)	3,997 (0.000)	3,797 (0.000)

For each multivariate CPH regression, this table displays the number of observations, estimates and robust standard errors (discussed in section 1.5.1), which can be observed in parenthesis. I employed the default estimation method in R, namely Efron approximation. In addition, in the last row of the table I also exhibit the results of likelihood ratio (LR) tests, in particular test statistics and p-values (in parenthesis). In line with standard notations, I denote estimates that are significant for 1 %, 5 % and 10 % significance levels as *, ** and ***, respectively. Since all PD variables are observed to be statistically significant at the 5 % level, we cannot reject assumption of structural credit risk models' importance. However, since, for example, NI/TA is statistically significant as well, we reject the null hypothesis that structural credit risk models are sufficient predictors.

Source: own work.

Table 8: Comparison of CPH regressions of time to default on structural credit risk model estimates and other explanatory variables for sub period 2001–2009.

Dependent Variable: Days to Default

Regression equation: $\log h_i(t) = \alpha(t) + x_i(t)'\beta$

Independent Variables	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7
π^{Merton}	1.630*** (0.414)						
$\pi^{Merton}_{\mu=r}$		2.519** (1.029)					
π^{Naive}			1.817*** (0.338)				
π^{BC}				1.580** (0.690)			
$\pi^{BC}_{Cons\,T}$					1.657** (0.694)		
π^{LS}						1.669** (0.696)	
π^{CDG}							1.412** (0.643)
$ln(F)$	-0.092 (0.197)	-0.106 (0.186)	-0.121 (0.182)	-0.101 (0.186)	-0.107 (0.182)	-0.108 (0.182)	-0.093 (0.189)

Continues

Continued

$1/\sigma_E$	-0.584*** (0.207)	-0.576** (0.251)	-0.480*** (0.184)	-0.572** (0.253)	-0.549** (0.252)	-0.547** (0.252)	-0.612** (0.251)
NI/TA	-2.308*** (0.702)	-2.857*** (0.477)	-2.166*** (0.616)	-2.809*** (0.487)	-2.782*** (0.496)	-2.776*** (0.497)	-2.861*** (0.465)
N	19,036	19,067	19,052	19,067	19,067	19,067	19,067
LR test	1,238 (0.000)	1,248 (0.000)	1,407 (0.000)	1,238 (0.000)	1,269 (0.000)	1,272 (0.000)	1,206 (0.000)

For each multivariate CPH regression, this table displays the number of observations, estimates and robust standard errors (discussed in section 1.5.1), which can be observed in parenthesis. Only sub period 2001–2009 has been accounted for in these regressions. I employed the default estimation method in R, namely Efron approximation. In addition, in the last row of the table I also exhibit the results of likelihood ratio (LR) tests, in particular test statistics and p-values (in parenthesis). In line with standard notations, I denote estimates that are significant for 1 %, 5 % and 10 % significance levels as *, ** and ***, respectively. Since all PD variables are observed to be statistically significant at the 5 % level, we cannot reject assumption of structural credit risk models' importance. However, since, for example, NI/TA is statistically significant as well, we reject the null hypothesis that structural credit risk models are sufficient predictors.

Source: own work.

Table 9: Comparison of CPH regressions of time to default on structural credit default model estimates and other explanatory variables including CDS implied PD.

Dependent Variable: Days to Default

Regression equation: $\log h_i(t) = \alpha(t) + x_i(t)'\beta$

Independent Variables	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7
π^{Merton}	-4.236*** (0.556)						
$\pi^{Merton}_{\mu=r}$		-7.784*** (1.613)					
π^{Naive}			-1.263*** (0.235)				
π^{BC}				-2.234*** (0.610)			
$\pi^{BC}_{Cons\,T}$					-2.010*** (0.625)		
π^{LS}						-2.018*** (0.624)	
π^{CDG}							-4.231*** (0.696)
π^{CDS}	13.657*** (1.625)	16.277*** (1.959)	9.837*** (1.418)	11.258*** (1.660)	10.985*** (1.677)	11.000*** (1.675)	14.156*** (1.750)
$\ln(E_t)$	-0.673*** (0.183)	-0.592*** (0.178)	-0.636*** (0.177)	-0.639*** (0.173)	-0.631*** (0.171)	-0.631*** (0.171)	-0.493*** (0.177)
$1/\sigma_E$	-1.166*** (0.152)	-1.325*** (0.219)	-0.873*** (0.091)	-1.064*** (0.093)	-1.034*** (0.090)	-1.033*** (0.089)	-1.204*** (0.129)

Continues

Continued

NI/TA	1.977 (3.688)	7.939*** (2.797)	4.291 (2.990)	5.900** (2.858)	5.805** (2.871)	5.791** (2.871)	4.514* (2.305)
N	9,035	9,040	9,040	9,040	9,040	9,040	9,040
LR test	380 (0.000)	378 (0.000)	346 (0.000)	350 (0.000)	348 (0.000)	348 (0.000)	381 (0.000)

For each multivariate CPH regression, this table displays the number of observations, estimates and robust standard errors (discussed in section 1.5.1), which can be observed in parenthesis. I employed the default estimation method in R, namely Efron approximation. In addition, in the last row of the table I also exhibit the results of likelihood ratio (LR) tests, in particular test statistics and p-values (in parenthesis). In line with standard notations, I denote estimates that are significant for 1 %, 5 % and 10 % significance levels as *, ** and ***, respectively.

Source: own work.

From the results in Table 7 and Table 8 one can conclude that several additional sets of data are relevant when building a model, which attempts to estimate time to default. That said, the Merton DD, BC, LS and CDG models cannot incorporate NI/TA which can result in a worse predictive power of structural credit risk model in comparison to the regression models presented in the above two tables. What is more, several additional independent variables for each of the 14 models are statistically significant at the 1 % significance level. Consequently, the results clearly suggest that purely relying only on any kind of structural credit risk model estimate can be deemed to be at least insufficient. Therefore, the hypothesis that structural credit risk models are sufficient predictors of default can be rejected, whereas their importance cannot be.

Apart from that, in Table 9 it can be noticed that all structural credit risk model predictors remained significant in spite of the CDS implied PD presence. As a result, one could claim that the relevance and importance of structural credit risk models is further confirmed. However, it has to be observed that the signs of estimates turned negative, which is non-intuitive as the PDs should be positively correlated with the hazard rate of default. Therefore, as the sign turned from intuitive in the univariate regression to non-intuitive, the usual suspect for such behavior is multicollinearity, even though the VIF did not manage to generally detect it. Indeed, the CDS implied PDs are relatively highly correlated with structural credit risk model's PDs as presented in Table 3. Hence, the values in Table 9 should not be interpreted and thus, the conclusion related to the importance of structural credit risk models cannot be further backed up through the results in Table 9. Nevertheless, the importance cannot be rejected either.

Generally, one could be tempted to compare the goodness of fit among the models by using methods such as the likelihood ratio (LR) test. However, note that not even two of the models in Table 7 and Table 8 are nested and hence, such test cannot be applied. By using other approaches, I will compare the models in the following section.

3.4 Contribution of the Structural Credit Risk Model's Complexity

In this section I attempt to examine whether the complexity of structural credit risk models adds any value to the PD estimation and time to default forecasts. On one hand, I will be interested in comparison of the Merton model types, namely π^{Merton}, $\pi^{Merton}_{\mu=r}$ and π^{Naive}, and the BC model types π^{BC} and $\pi^{BC}_{Cons\,T}$ on the other. More importantly, the main comparison will be between the directly comparable $\pi^{Merton}_{\mu=r}$, $\pi^{BC}_{Cons\,T}$, π^{LS} and π^{CDG}.

First of all, I directly compare the quantiles for all data points for which the data set is complete, meaning that all PD estimates, including the CDS implied, have been calculated. There are 8,384 such observations, which may appear as rather low. In Table 10 I display means, standard deviation and quantiles. It can be seen that the Naïve model importantly overestimates the PD, whereas the CDG, LS and BC models are closer in line with the CDS implied PD. Nonetheless, it is easy to observe that all structural credit risk models overestimate the PD of high-risk companies with respect to the market but underestimate the PD of the safer ones. This is consistent with Huang and Huang (2002), and Suo and Wang (2006) notes.

Table 10: Summary statistics of PD estimates for data points with complete data set.

				Quantiles				
Variable notation	N	Mean	Std. Dev.	Min	25 %	Median	75 %	Max
π^{Merton}	8,384	3.17	12.57	0.00	0.00	0.00	0.01	99.91
$\pi^{Merton}_{\mu=r}$	8,384	1.41	6.12	0.00	0.00	0.00	0.01	78.76
π^{Naive}	8,384	5.99	19.04	0.00	0.00	0.00	0.09	100.00
π^{BC}	8,384	2.54	10.64	0.00	0.00	0.00	0.02	100.00
$\pi^{BC}_{Cons\,T}$	8,384	2.65	11.10	0.00	0.00	0.00	0.02	100.00
π^{LS}	8,384	2.65	11.09	0.00	0.00	0.00	0.02	100.00
π^{CDG}	8,384	1.88	9.57	0.00	0.00	0.00	0.00	100.00
π^{CDS}	8,384	1.95	3.39	0.09	0.56	0.95	1.92	70.33

This table reports summary statistics, including quantiles, means, standard deviations and number of observations for each variation of the analyzed structural credit risk models. All values are expressed in percentages, except for the number of observations.

Source: own work.

PD estimates can also be benchmarked against the actual default rate of companies in the sample. For each observation I evaluated whether the default of a company occurred in 1 year or less. Thus, I obtained actual default rate values as presented in equation (68).

$$Actual\ Default = \begin{cases} 1; if\ defaulted\ in\ \leq 1\ year \\ 0; if\ not\ defaulted\ in \leq 1\ year \end{cases} \tag{68}$$

Therefore, Table 11 suggests that models generally overestimate actual PD. It turns out that the least reasonable values are produced by the Naïve model as average PD is substantially higher than actual default rates and other PD estimates. However, neither Table 10 nor Table 11 suggest that PD estimates would be closer to the market or actual default rates for more complex structural credit risk models, namely the BC, LS and CDG models.

Table 11: Average actual default rate benchmarked against means of structural credit risk models PD estimates.

Average actual default rate	π^{Merton}	$\pi^{Merton}_{\mu=r}$	π^{Naive}	π^{BC}	$\pi^{BC}_{Cons\,T}$	π^{LS}	π^{CDG}
0.62	4.10	2.25	6.60	3.87	4.09	4.09	3.26

This table compares means of PD estimates for each variation of analyzed structural credit risk models with the average actual default rate. The analysis has been performed on 51,223 observations as some were dropped due to actual default data censorship within the last year. All values are expressed in percentages.

Source: own work.

In the following part, I examine the contribution of more complex models based on multivariate CPH regressions displayed in Table 12. In order to verify interpretability, I confirmed that the VIF values do not exceed 4.13 for any of the 5 models below, which is acceptable.

The most meaningful results can be observed in Models 2 and 3. In particular, in Model 2, the π^{Merton} predictor turns out to be statistically insignificant when regressing it along with the π^{Naive} estimator, which remains significant at all 3 significance levels. As a result, the results are in line with the observation in Bharath and Shumway (2008). They stated that the calibration of asset volatility, asset return and market asset value in the Merton DD model appears to be unimportant given the outcome of this test. Therefore, it can be claimed that all predictive power of the Merton model lies in the functional form which has been maintained in the Naïve model as well.

Apart from it, Model 3 suggests that the incremental value of the Longstaff-Schwartz model PD estimate with respect to the Black-Cox PD estimate remains statistically insignificant at the 1 % and 5 % confidence levels. Therefore, even though I concluded in sections 3.2 and 3.3 that individual $\pi^{BC}_{Cons\,T}$ and π^{LS} may be important predictors, the results suggest that the incremental value of the LS model does not add any value given a 5 % confidence level. As a result, at least based on my data set and Vasicek parameters calibration used, it appears that the implementation of the Vasicek model and the Longstaff-Schwartz PD approximation presented in section 1.3.2 do not contribute to a better prediction of the time to default. Indeed, one may expect it based on the comparison of PDs in Figure 9. In a similar fashion, Table 3 shows that the quantiles and averages are very closely in line, whereas the correlation between the two is close to 1. On the other hand, $\pi^{BC}_{Cons\,T} - \pi^{BC}$ and $\pi^{LS} - \pi^{CDG}$ are statistically significant in Models 4 and 5, respectively; hence, I cannot conclude that the contribution of the CDG model is negligible. Similarly, the modelling choice in the BC model, regarding how to build a barrier, may be important for the purposes of time to default forecasting.

Table 12: Comparison of Cox proportional hazard regressions of time to default on PD estimates and differences between estimates.

Dependent Variable: Days to Default					
Regression equation: $\log h_i(t) = \alpha(t) + x_i(t)'\beta$					
Independent Variables	Model 1	Model 2	Model 3	Model 4	Model 5
π^{Merton}		0.528 (0.510)			
$\pi^{Merton}_{\mu=r}$	1.699*** (0.655)				
π^{Naive}	1.790*** (0.220)	1.968*** (0.377)			
π^{BC}				2.370*** (0.304)	
$\pi^{BC}_{Cons\,T}$			2.547*** (0.326)		
π^{CDG}					2.263*** (0.349)
$\pi^{LS} - \pi^{BC}_{Cons\,T}$			21.595* (11.215)		
$\pi^{BC}_{Cons\,T} - \pi^{BC}$				5.404*** (1.770)	
$\pi^{LS} - \pi^{CDG}$					5.532*** (0.728)

Continues

Continued

N	55,671	53,355	55,688	55,688	55,688
LR test	2,462 (0.000)	2,203 (0.000)	1,844 (0.000)	1,859 (0.000)	2,167 (0.000)

In this table I examine the contribution of more complex models through results of multivariate CPH regressions. For each CPH regression, this table displays the number of observations, estimates and robust standard errors (discussed in section 1.5.1), which can be observed in parenthesis. I employed the default estimation method in R, namely Efron approximation. In addition, in the last row of the table I also exhibit the results of likelihood ratio (LR) tests, in particular test statistics and p-values (in parenthesis). In line with standard notations, I denote estimates that are significant for 1 %, 5 % and 10 % significance levels as *, ** and ***, respectively. Since difference between the LS and CDG PD is not statistically significant at the 5 % level, we can claim that the LS model does not significantly improve the BC model's performance.

Source: own work.

Next, I compare all 7 analyzed structural credit risk model variations against the actual default data. This can be achieved through backtesting. In a similar manner as Bharath and Shumway (2008) compared the predictive powers of the Naïve and Merton DD model, I assess the contribution of complexity in structural credit risk modelling. For each date I distribute companies into deciles, based on the ranking of their PD estimates, meaning that the firms in decile 1 are the riskiest, whereas those in decile 10 are the safest given the model's framework. Such assessment may not fully reflect the model but it still accounts for the classification of companies. Bharath and Shumway (2008) claim that the first of the two main advantages of such analysis is that no actual default probabilities are needed as a benchmark. In addition, in section 1.1.2 I mention that one of the main differences between Merton DD model and KMV model is that KMV relies on the empirical distribution of distances to default when converting them into a PD. On the other hand, researchers do not possess such database so they have to rely on normal distribution even though it may not be the ideal choice. However, in the analysis below, this conversion is not relevant as only the classification against other companies matters.

Ideally, the highest possible amount of companies that are about to default within the upcoming 1-year period, should drop into decile 1. On the other hand, the percentage of companies, which are classified as the safest ones but still default in less than 1 year, should be as low as possible. Table 13 summarizes the results of the analysis which has been performed on 318 observations for which companies were less than 1 year away from the default. One can notice that, surprisingly, the Naïve model ranked the most of such companies among decile 1, more than 5% above the rest of the models. Furthermore, the most complex CDG model performed marginally worse than $\pi_{\mu=r}^{Merton}$, the BC model variations and LS model, in the classification of almost defaulted companies in decile 1. Nonetheless, generally it can be seen than $\pi_{\mu=r}^{Merton}$, π^{BC}, $\pi_{Cons\,T}^{BC}$ and π^{LS} indicate a very similar performance in the analyzed dataset of defaulted companies. Figure 11 displays the results already presented in Table 13. It indicates that none of the models dominates other ones, meaning that none of the lines is above the rest for each of the 10 deciles. It can be observed though that the Naïve model dominates the Merton DD model for most of the deciles, which indicates that the Merton DD classifies companies worse than its simplified version.

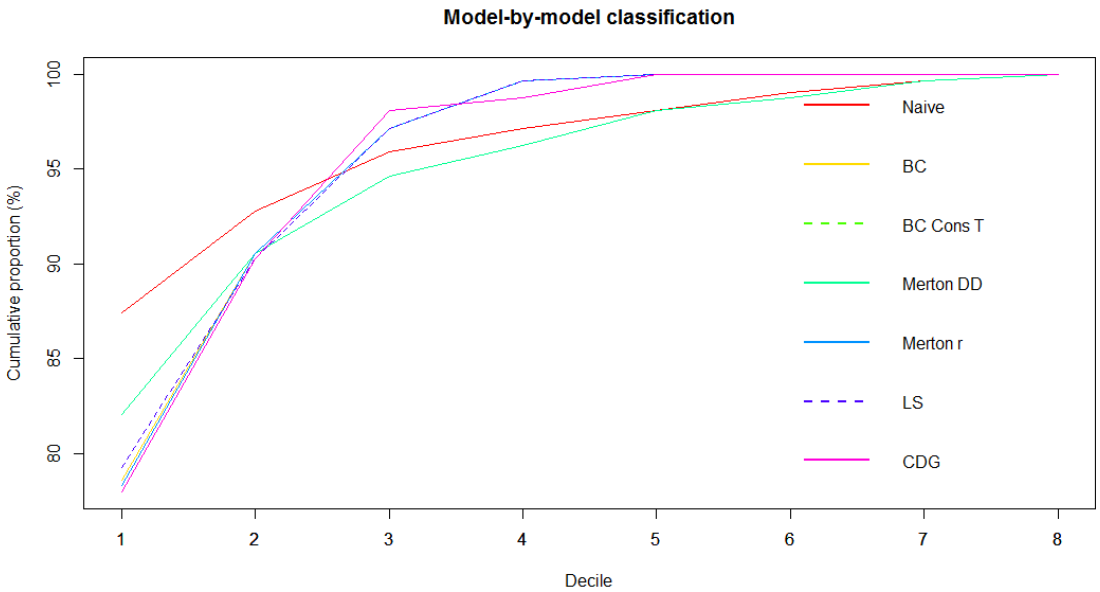
Table 13: Classification of companies that defaulted in less than a year into deciles.

Deciles	π^{Merton}	$\pi^{Merton}_{\mu=r}$	π^{Naive}	π^{BC}	$\pi^{BC}_{Cons\,T}$	π^{LS}	π^{CDG}
1	82.08	78.30	87.42	78.62	79.25	79.25	77.99
2	8.49	12.26	5.35	11.95	11.01	11.01	12.26
3	4.09	6.60	3.14	6.60	6.92	6.92	7.86
4	1.57	2.52	1.26	2.52	2.52	2.52	0.63
5	1.89	0.31	0.94	0.31	0.31	0.31	1.26
6	0.63	0.00	0.94	0.00	0.00	0.00	0.00
7	0.94	0.00	0.63	0.00	0.00	0.00	0.00
8	0.31	0.00	0.31	0.00	0.00	0.00	0.00

This table summarizes results of backtesting performed on 318 observations for which companies were less than 1 year away from the default. For each date I distribute companies into deciles, based on the ranking of their PD estimates, meaning that firms in decile 1 are the riskiest, whereas those in decile 10 are the safest given the model's framework. Thus, I report fraction of companies that have been allocated in a given decile. Results are presented in percentages.

Source: own work.

Figure 11: Cumulative proportions of companies classified in a given decile or lower.



Source: own work.

Finally, the ultimate aim of such PD models is to place the PD close to 0 if the company is not going to default in less than a year and 1 if so. Such benchmark variable can be expressed as in the equation (68). The lower the deviation of model's PD estimates from such variable is, the more accurate the model is deemed to be.

Table 14 reports the results of such accuracy study which has been conducted on 51,223 observations where the MSE (mean squared error) metric has been considered. Note that the data between August 2016 and August 2017 had to be removed due to censorship. The results illustrate that the Naïve model performs the worst, whereas the $\pi_{\mu=r}^{Merton}$ surprisingly performs the best with the MSE reaching only 0.97 %. Overall, there is no ultimate evidence that more complex structural credit risk models would outperform the simpler ones. Nonetheless, one has to note that this conclusion may be a result of the analysis being performed on a sample of companies which is considered to be a relatively low default portfolio.

Table 14: Comparison of models' accuracy based on benchmarking the actual default variable against PD estimates.

Model	π^{Merton}	$\pi_{\mu=r}^{Merton}$	π^{Naive}	π^{BC}	$\pi_{Cons\,T}^{BC}$	π^{LS}	π^{CDG}
MSE (%)	2.21	0.97	4.21	2.05	2.24	2.24	1.93

This table reports a comparison of mean squared errors per structural credit risk model with respect to the actual 1-year default variable. The study has been conducted on 51,223 observations and the MSE accuracy measure has been used. All values are expressed in percentages.

Source: own work.

CONCLUSIONS

I examine four structural credit risk models and their seven variations. By doing so, I attempt to answer the questions that are specified in the introduction of the paper. Firstly, the results in section 3.1 suggest that it cannot be claimed that any model is generally less sensitive to bumps of any of the main parameters. For example, the asset market value sensitivity in the CDG model is the highest for the companies close to default but lower for the less risky companies. The Merton model sensitivities are lower when the leverage is above one but due to its construction it remains sensitive even for the companies with negative capital. Nonetheless, the LS and CDG models also embed sensitivities against the Vasicek parameters, whereas the CDG accounts for sensitivity with respect to the barrier parameters too. As a result, the LS and CDG models' simplicity is affected. Besides, they are more dependent on the modelling choices that a practitioner or researcher has to make when calibrating them.

Secondly, in sections 3.2 and 3.3 I investigate whether models are important and whether they are sufficient. In a similar manner as the research performed by Bharath and Shumway (2008) on Merton model variations, I observe that all structural credit risk models are relevant and possess predictive power when forecasting the time to default. Indeed, this observation is maintained even when including other explanatory variables into the CPH regression, which indicates the importance of the estimates. However, when including the CDS implied default probabilities in the CPH regression, multicollinearity has been detected. Hence, as opposed to Bharath and Shumway (2008), the importance cannot be further backed up but it cannot be rejected either. Next, when including more parameters into regressions I note that several other independent variables are statistically significant which means that none of the analyzed structural credit risk models is a sufficient predictor. More importantly, since the statistically significant variables are also those that structural credit risk models do not account for, such as excess return or NI/TA, one can fundamentally question the construction of these models.

Thirdly, in section 3.4 I aim to compare the models and assess whether any of them indicates better performance than the others. When comparing the average PD estimates against the market implied PD in Table 10, one can observe that the Merton DD and especially the Naïve models in average overestimate the PD. What is more, all structural credit risk models appear to underestimate the riskiness of safer companies whereas the riskiness of the firms with a higher PD is further amplified within such structural model type framework.

Moreover, when comparing the average PD on the full set of data against the average actual 1-year actual default rate, it can be observed that all models overestimate the default, with the Naïve model appearing to be the worst. On the other hand, I perform regressions with two structural credit risk models being among explanatory variables. Surprisingly, the CPH model reports that the Merton DD model turns insignificant when regressing it along with the Naïve model, which indicates that the Naïve model contributes more to the explanation of the time to default. In a similar manner, the outcome of regressions confirms that the LS model does not improve the forecasting power with respect to the BC model with a constant threshold given the 5 % confidence level. During backtesting, when classifying companies into deciles from the riskiest to the safest, the Naïve model outperforms the Merton DD model, whereas the more complex models do not dominate any of the Merton model's versions. Nonetheless, one should note that this conclusion may be a result of the analysis being performed on a sample of companies which is considered to be a relatively low default portfolio – more specifically on S&P 500 companies.

As a result of the above, I conclude that given my dataset there is no clear evidence that more complex models would significantly contribute to the forecasting performance or to a better classification of companies based on their riskiness. On the other hand, it can be once again observed, similarly as in Bharath and Shumway (2008), that the Merton model and other structural credit risk model may be important in the time to default estimation. This has been confirmed when including additional independent variables in the CPH regressions. On the other hand, the importance cannot be further confirmed through regressions, where the CDS implied PDs are one of the regressors, due to presence of multicollinearity.

The Merton DD model's usefulness comes from the functional form of the model as it cannot be claimed that the Naïve model generally underperforms. Apart from it, practice confirms theory and Merton's PD estimates under the risk-neutral measure are indeed always lower than the BC predictions. As a result, institutions and other investors that use the BC model rather than the Merton model in order to assess PD are always on the conservative side, whereas the Merton model turns out to be the aggressive choice. Finally, it has to be pointed out that even though the dataset presented in section 2 covers more than 53,000 data points for most of the variables, the increased scope of companies could further benefit my research.

Overall, the structural credit risk models offer interpretability that some other types of PD models may not. Nonetheless, the performance of such models is limited due to their construction as several other relevant predictors are not included in the model. Besides, based on my low default portfolio there is no proof that the increased complexity of structural credit risk models, which incorporate the stochastic modelling of interest rates and the stochastic default barrier, can significantly improve it. However, the usefulness of structural credit risk models in the time to default forecasting may improve when using them as independent variables in regressions along with other relevant predictors.

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APPENDICES

Appendix 1: Povzetek magistrskega dela (Summary of the Thesis in Slovenian language)

Napovedovanje verjetnosti neizpolnitve (angl. probability of default; v nadaljevanju PD) je dandanes ena najpomembnejših nalog v finančnem sektorju. PD modeli med drugim omogočajo natančnejšo kreditno razvrstitev (angl. credit ranking) podjetij glede na tveganost, obenem pa odločajo tudi o tem, ali so krediti podeljeni ali ne. Alternativo tradicionalnim regresijskim PD modelom predstavljajo strukturni modeli kreditnega tveganja. Slednji so glavna tema magistrskega dela, kjer predstavim in primerjam Mertonov, Black-Coxov (v nadaljevanju BC), Longstaff-Schwartzev (v nadaljevanju LS) ter Collin-Dufresne in Goldsteinov (v nadaljevanju CDG) model.

Strukturni modeli kreditnega tveganja predstavljajo tip PD modelov, ki uporabijo idejo vrednotenja opcij. Strukturni modeli prezrcalijo verjetnost neizvršitve, ki je pomembna v okvirih vrednotenja opcij, v PD. Prvi model v svetu strukturnih kreditnih modelov, Mertonov (1974) model, vidi tržno vrednost podjetja kot evropsko nakupno opcijo na tržno vrednost svojih naložb. Black in Cox (1976) opažata, da Merton (1974) nerazumno predpostavi, da vse obveznosti podjetja zapadejo hkrati. Posledično razvijeta BC model, ki vidi tržno vrednost podjetja kot nakupno opcijo z mejo, kar pomeni, da podjetje lahko propade že pred iztekom napovednega obdobja. V nadaljevanju Longstaff in Schwarz (1995) odpravita predpostavko o konstantni ne tvegani obrestni meri, medtem ko Collin-Dufresne in Goldstein (2001) zastavita svoj model tako, da postane Black-Coxova meja stohastična. Le-ta je odvisna od vrednosti naložb in tudi od ne tvegane obrestne mere.

Glavna raziskovalna vprašanja:

1. Kako občutljivi so strukturni modeli kreditnega tveganja na spremembe svojih vhodnih podatkov in kako se občutljivosti razlikujejo med modeli?
2. Ali je kateri izmed strukturnih modelov kreditnega tveganja zadostna ali vsaj pomembna cenilka časa do neizpolnitve?
3. Ali kompleksnost strukturnih modelov kreditnega tveganja doprinese k natančnosti ocenjevanja časa do neizpolnitve in če da, koliko?

V uvodu je poudarjena pomembnost modeliranja kreditnih tveganj, opisani so tudi tipi PD modelov. Poleg tega so predstavljene ključne ideje najbolj znanih strukturnih modelov kreditnega tveganja, povzeti so tudi zaključki nekaterih študij in njihova dognanja. Prav tako so omenjeni ključni cilji magistrskega dela in morebitni prispevek k že obstoječi literaturi.

Prvo poglavje se osredinja na matematično ozadje prej omenjenih strukturnih modelov kreditnega tveganja, njihove predpostavke ter metode kalibracije parametrov, kot na primer Vasičkovih parametrov. Poleg tega so omenjene razloge in logika posameznega pristopa strukturnega modeliranja kreditnih modelov. V nadaljevanju so predstavljene še ideje Coxovega proporcionalnega hazardnega modela (angl. Cox proportional hazard model), ki ga uporabim kot glavno metodo določanja, ali je model pomemben in ali je zadosten za napovedovanje časa do neizpolnitve. Nazadnje je prikazan še poenostavljen model za vrednotenje zamenjave kreditne neizpolnitve (angl. credit default swap; v nadaljevanju CDS). Ta produkt je še posebej relevanten, saj tržna cena takšnega finančnega produkta vsebuje podatek, kako trg ocenjuje verjetnost neizpolnitve za izbrano podjetje.

Drugo poglavje opiše vhodne podatke, njihove vire, kalibrirane spremenljivke ter primerja vzorčna povprečja, vzorčne standardne odklone in kvantile ocen PD-ja za vsak strukturni model kreditnega tveganja. Na tem mestu potrdim teorijo, ki zagotavlja, da BC model vedno ocenjuje PD višje kot Mertonov model. Tako lahko trdim, da je BC model vedno konservativnejša izbira za finančne institucije. Prav tako predstavim, kateri dodatni podatki bi bili potrebni v primeru razširjene študije. Tako bi med drugim lahko bila uporabljena volatilnost, implicirana iz tržnih cen opcij.

Podpoglavje 3.1 se osredotoči na analizo občutljivosti PD ocen glede na posamezne spremembe v vhodnih parametrih. Tu tudi medsebojno primerjam občutljivosti strukturnih modelov kreditnega tveganja in razpravljam o razlogih za opažana obnašanja modelov. V zadnjem vsebinskem poglavju so prikazani in interpretirani rezultati empirične analize, v zaključku pa so podani odgovori na ključna prej omenjena vprašanja, ki so zastavljena v uvodu dela.

Rezultati te študije kažejo na to, da ne morem trditi, da je kateri izmed modelov manj občutljiv na spremembe glavnih parametrov. A je treba vseeno poudariti, da LS in CDG modela dodatno kalibrirata Vasičkove parametre, medtem ko CDG model vsebuje tudi parametre, ki vplivajo na obveznosti podjetja. Posledično sta zadnja dva modela občutljiva na večjo količino parametrov, kar sovpada s tem, da sta LS in CDG modela manj preprosta za interpretacijo.

V podpoglavjih 3.2 in 3.3 razpravljam o tem, ali je kateri izmed modelov pomemben oziroma zadosten pri ocenjevanju časa do neizpolnitve. Izidi testov razkrivajo, da ne moremo trditi, da so štirje strukturni modeli nepomembni, gotovo pa lahko rečemo, da niso zadostni. Poleg tega se izkaže, da so dodatni parametri, kot na primer kvocient neto prihodkov ter naložb podjetja, prav tako pomembni ocenjevalci časa do neizpolnitve. Ker pa nobeden izmed strukturnih modelov kreditnega tveganja teh informacij ne upošteva, lahko sklenem, da je učinkovitost modelov vprašljiva že v njihovi zasnovi.

Nazadnje primerjam vse omenjene modele kreditnega tveganja in poskušam ugotoviti, ali kateri izkazuje boljšo uspešnost pri napovedovanju časa do neizpolnitve. Najprej primerjam povprečja ter kvantile generiranih ocen PD-ja, kjer opazim, da vsi modeli podcenjujejo tveganost varnejših podjetij, obenem pa dodatno precenjujejo PD-je bolj tveganih podjetij v primerjavi s trgom. Poleg tega rezultati analiz nakazujejo na to, da se LS model značilno ne razlikuje ter da ne izboljša napovedi časa do neizpolnitve v primerjavi z BC modelom s konstantno mejo. Posledično lahko trdim, da kalibracija Vasičkovih parametrov na našem vzorcu ne pripomore k boljšim napovedim.

Empirično analizo sklenem tako, da na zgodovinskih podatkih preverim, kako uspešno modeli razvrščajo podjetja v decile od manj do bolj tveganih. Rezultati kažejo na to, da naivna verzija Mertonovega modela dominira Mertonovo izpeljanko, to je Mertonov DD model, ki vključuje kalibracije vrednosti naložb, njihovo volatilnost ter donose. Po drugi strani pa je mogoče opaziti, da se kompleksnejši modeli na splošno ne odrežejo znatno bolje v primerjavi z manj kompleksnimi strukturnimi modeli, konkretneje z Mertonovimi izpeljankami. Do podobnega zaključka pridem, ko primerjam natančnosti napovedovanja z povprečji kvadratov napak – ni dokaza, da bi kompleksnejši modeli dominirali manj kompleksne strukturne modele kreditnega tveganja. Tu je sicer potrebno poudariti, da razlog morda tiči v tem, da je moj portfelj podjetij načeloma manj tvegan, zato je pričakovano, da se Mertonov model, ki sistematično ocenjuje PD nekoliko nižje kot recimo Black-Cox model, bolje prilega realizacijam.

Posledično lahko zaključim, da strukturni modeli sicer ponujajo interpretacijo, ki je nekateri drugi tipi PD modelov morda ne. A je uspešnost teh modelov omejena in glede na uporabljeni vzorec podjetij ni jasnega dokaza, da bi kompleksnejši strukturni modeli kreditnega tveganja to znatno spremenili. Uporabnost strukturnih modelov za napovedovanje časa do neizpolnitve se izboljša, ko jih v regresiji uporabimo kot neodvisne spremenljivke skupaj z drugimi značilnimi spremenljivkami.

Appendix 2: Processes under T-forward measure in LS model

The dynamics of the risk-free rate under the T-forward measure in the LS model is presented in equation (69)

$$dr_t = \beta\left(\theta - r_t - \frac{\eta^2}{\beta}B(T-t,\beta)\right)dt + \eta dW_{r,t}^T \qquad (69)$$

where

$$B(s,\beta) = \frac{1}{\beta}\left(1 - e^{-\beta s}\right) \qquad (70)$$

On the other hand, log asset value process is expressed with the following formula where function B is given in (70). Note that the log-process of market asset value is denoted as $y_t = logV_t$ (Longstaff & Schwartz, 1995).

$$dy_t = (r_t - \frac{\sigma_V^2}{2} - \rho\sigma_V\eta B(T-t,\beta))dt + \sigma_V dW_t \tag{71}$$

Appendix 3: Conditional Moments of Variables in CDG model

Collin-Dufresne and Goldstein (2001) defined log-default boundary l_t and risk-free rate r_t of their own model as discussed in section 1.4. Likewise for the special case of their model, LS model, Collin-Dufresne and Goldstein (2001) also presented implementation of the PD approximation under the CDG framework. They derived exact solutions of expected value and variance according to both models under the T-forward measure. Risk-free rate processes for the two models are identical and presented in formula (69). Nonetheless, due to the non-constant obligations, leverage process l_t is not equivalent to LS model but follows:

$$dl_t = \lambda\left(\bar{l}^Q - \frac{1+\lambda\phi}{\lambda}r_t - l_t + \frac{\rho\sigma_V\eta}{\lambda}B(T-t,\beta)\right)dt - \sigma_V dW_t^T \tag{72}$$

where

$$\bar{l}^Q = \frac{\sigma_V^2}{2\lambda} - \nu + \phi\theta \tag{73}$$

Moreover, in the CDG model expected value and variance, respectively, of the leverage process l_t equal:

$$\begin{aligned} E_s^T(l_t) = {} & l_s e^{-\lambda(t-s)} - (1+\lambda\phi)\left(r_u + \frac{\eta^2}{\beta^2} - \theta\right)e^{-\beta(t-s)}B(t-s,\lambda-\beta) \\ & - \left(\frac{\rho\sigma_V\eta}{\beta} + (1+\lambda\phi)\frac{\eta^2}{2\beta^2}\right)e^{-\beta(t-s)}B(t-s,\lambda+\beta) \\ & + (1+\lambda\phi)\frac{\eta^2}{2\beta^2}e^{-\beta(T-t)}e^{-2\beta(t-s)}B(t-s,\lambda-\beta) \\ & + \left(\frac{\rho\sigma_V\eta}{\beta} + \lambda\bar{l}^Q - (1+\lambda\phi)\left(\theta - \frac{\eta^2}{\beta^2}\right)\right)B(t-s,\lambda) \end{aligned} \tag{74}$$

and

$$
\begin{aligned}
Var_s^T(l_t) = {} & \left(\frac{(1+\lambda\phi)\eta}{\lambda-\beta}\right)^2 B(t-s,2\beta) \\
& + \left(\sigma_V^2 + \left(\frac{(1+\lambda\phi)\eta}{\lambda-\beta}\right)^2 - 2\frac{\rho\sigma_V(1+\lambda\phi)\eta}{\lambda-\beta}\right) B(t-s,2\lambda) \\
& + 2\left(\frac{\rho\sigma_V(1+\lambda\phi)\eta}{\lambda-\beta} - \left(\frac{(1+\lambda\phi)\eta}{\lambda-\beta}\right)^2\right) B(t-s,\lambda+\beta)
\end{aligned} \tag{75}
$$

On the other hand, since the SDE of risk-free rate process remains untouched with respect to the LS model, the risk-free rate's expected value and variance remain the same as well, meaning that:

$$
E_s^T(r_t) = r_s e^{-\beta(t-s)} + \left(\theta\beta - \frac{\eta^2}{\beta}\right) B(t-s,\beta) + \frac{\eta^2}{\beta} e^{-\beta(T-t)} B(t-s,2\beta) \tag{76}
$$

$$
Var_s^T(r_t) = \eta^2 B(t-s,2\beta) \tag{77}
$$

Finally, the covariance between the two processes is determined by:

$$
\begin{aligned}
Cov_s^T(l_t,r_t) = {} & -\frac{(1+\lambda\phi)\eta^2}{\lambda-\beta} B(t-s,2\beta) \\
& - \left(\rho\sigma_V\eta - \frac{(1+\lambda\phi)\eta^2}{\lambda-\beta}\right) B(t-s,\lambda+\beta)
\end{aligned} \tag{78}
$$

Appendix 4: CDG model – R code implementation

```
## function B is equivalent to function B in section 1.4

B <- function(s, beta) {return(1/ beta * (1 - exp(-beta * s)))}

## Fi function is a generalization and computes both Fi and Psi as in equations (52) and (53), respectively

Fi <- function(beta, theta, sigmaR, Fs, Vs, t, s, sigmaV, rho, rs, Tmat, rt, fi, lambda, ni){
  # beta..Vasicek parameter (intensity)
  # theta .. Vasicek parameter (long-term)
  # sigmaR .. Vasicek vol
  # Fs .. barrier
  # Vs .. assets
  # t .. time t
  # s .. time s (t>=s)
  # sigmaV ... asset vols
  # rho .. correlation
  # rs .. interest rate at time s
```

```
  # Tmat ... maturity
  # rt ... hypothetical interest rate at time t
  # fi ... CDG parameter of interest rate influence on debt
  # lambda ... CDG parameter of influence size of assets (with corrections) on debt
  # ni ... CDG target log leverage ratio

  k <- log(Fs) #log barrier
  ys <- log(Vs)
  ls <- k - ys
  lQ <- (sigmaV^2 / 2) / lambda - ni + fi * theta

  ## moments calculation
  E_s_l <- ls *exp(- lambda * (t - s)) - (1 + lambda * fi) * (rs + sigmaR^2/ beta ^2 - theta) *
exp(-beta * (t - s)) * B(t-s, lambda - beta) -
    (sigmaR * sigmaV * rho/ beta + (1 + lambda * fi) * sigmaR^2/(2 * beta ^2)) * exp(-beta
* (Tmat - t)) * B(t-s, lambda + beta)  +
    (1 + lambda * fi) * sigmaR^2/(2 * beta ^2) * exp(-beta * (Tmat - t)) * exp(- 2 * beta * (t-
s)) * B(t-s, lambda - beta)  +
    (sigmaR * sigmaV * rho/ beta + lambda * lQ - (1 + lambda * fi) * (theta - sigmaR^2/ beta
^2)) * B(t - s, lambda)

  Var_s_l <- ((1 + lambda * fi) * sigmaR/(lambda - beta))^2 * B(t - s, 2 * beta) +
    (sigmaV^2 + ((1+ lambda * fi) * sigmaR/(lambda - beta))^2 - (2 * rho * sigmaV * (1 +
lambda * fi) * sigmaR/(lambda - beta))) * B(t - s, 2 * lambda) +
    2 * ((rho * sigmaV * (1 + lambda * fi) * sigmaR/(lambda - beta)) - ((1 + lambda * fi) *
sigmaR /(lambda - beta))^2) * B(t - s, lambda + beta)

  E_s_r <- rs * exp(-beta * (t - s)) + (theta * beta - (sigmaR^2)/ beta) * B(t - s, beta) +
(sigmaR^2)/ beta * exp(-beta * (Tmat - t)) * B(t - s, 2 * beta)

  Var_s_r <- sigmaR^2 * B(t - s, 2 * beta)

  Cov_s <- -(1+lambda*fi) * sigmaR^2/(lambda - beta) * B(t - s, 2 * beta) - (sigmaV * rho *
sigmaR - (1+lambda*fi) * sigmaR^2/(lambda - beta))* B(t - s, lambda + beta)

  E_lt <- E_s_l + (Cov_s^2)/Var_s_r * (rt - E_s_r)

  #sanity check that Var is positive
  Var_lt <- pmax(Var_s_l - (Cov_s^2)/Var_s_r, 0)
  # final solution for Fi/Psi (a
  return(pnorm(E_lt/sqrt(Var_lt)))
}

## transition probability function as in equation (56) and (57)

transProbs <- function(t, sigmaR, beta, theta, rt, r0){
  gamma <- sqrt((sigmaR^2) * (1 - exp(-2 * beta * t)))
```

```
  transProb <- 1/(sqrt(pi/ beta) * gamma) * exp(-((rt-theta - (r0-theta) * exp(-beta * (t)))^2)
* beta / gamma^2)

  return(transProb)
}

## a parent function that produces final results as in equation (46)

CDGprob <- function(beta, theta, sigmaR, Fs, Vs, sigmaV, rho, r0, Tmat, n_r, n_T, fi,
lambda, ni){
  # beta..Vasicek parameter (intensity)
  # theta .. Vasicek parameter (long-term)
  # sigmaR .. Vasicek vol
  # Fs .. barrier
  # Vs .. assets
  # sigmaV ... asset vols
  # rho .. correlation
  # r0 ... interest rate at time s
  # Tmat ... maturity
  # n_r ... number of r scenarios
  # n_T ... number of discretized intervals of time T
  # fi ... CDG parameter of interest rate influence on debt
  # lambda ... CDG parameter of influence size of assets (with corrections) on debt
  # ni ... CDG target log leverage ratio

  if(any(is.na(c(beta, theta, sigmaR, Fs, Vs, sigmaV, rho, r0, Tmat, n_r, n_T, fi, lambda,
ni)))){
    return(NA)

  }else{

    if(Fs>Vs){
      return(1)
    }else{
      r_dev <- 3 * sqrt((sigmaR^2)/(2* beta))
      r_low <- theta - r_dev

      delta_r <- 2 * r_dev/n_r
      delta_t <- Tmat/n_T

      r_all <- seq(r_low + delta_r, r_low + delta_r * n_r, delta_r)
      t_all <- seq(delta_t, delta_t * n_T, delta_t)

      # Vasicek transition probabilities between Markov states
      r_startArray <- array(r_all, dim = c(n_r, n_r, n_T))
      r_endArray <- aperm(array(r_all, dim = c(n_r, n_r, n_T)), c(2, 1, 3))
      t_Array <- aperm(array(t_all, dim = c(n_T, n_r, n_r)), c(2, 3, 1))
      transProbFinal <- transProbs(t_Array, sigmaR, beta, theta, r_endArray, r_startArray)
```

```
    # Vasicek transition probs in time t conditional to start (time 0)
    r_endArray2 <- array(r_all, dim = c(n_r, n_T))
    t_Array2 <- aperm(array(t_all, dim = c(n_T, n_r)), c(2, 1))
    transProbFinal2 <- transProbs(t_Array2, sigmaR, beta, theta, r_endArray2, r0)

    q <- matrix(0, n_r, n_T)
    r_sArray <- array(r_all, dim = c(n_r, n_T, n_r))
    r_tArray <- aperm(array(r_all, dim = c(n_r, n_T, n_r)), c(3, 2, 1))
    s <- t(matrix(t_all, nrow = n_T, ncol = n_r))

    for(j in 1:n_T){

      t <- t_all[j]

      if (j != 1){

        for(i in 1:n_r){
          # Fi * q
          q[i, j] <- sum(q[, 1:(j-1)] * Fi(beta, theta, sigmaR, Fs, Fs, t, s[, 1:(j-1)], sigmaV, rho,
r_all, Tmat, r_all[i], fi, lambda, ni) * transProbFinal[, i, (j-1):1])
        }

      }

      # add psi
      q[, j] <- delta_r * pmax(Fi(beta, theta, sigmaR, Fs, Vs, t, 0, sigmaV, rho, r0, Tmat, r_all,
fi, lambda, ni) * transProbFinal2[, j] - q[, j], 0)

    }

    return(sum(q))
  }

 }

}
```

Appendix 5: LS model – R code implementation

```
# equation (42)

S <- function(rho, sigmaV, alpha, beta, ni, t){
  return(((sigmaV * rho * ni)/beta + (ni/beta)^2+ sigmaV^2) * t - ((sigmaV * rho * ni)/beta^2
+ 2*(ni^2)/(beta^3)) * (1 - exp(- beta * t)) +
         ((ni^2)/(beta^3))/2 * (1 - exp(-2 * beta * t)))
}

# equation (41)
```

```
M <- function(rho, sigmaV, alpha, beta, ni, t, Tmat, r0){
  return(((alpha - sigmaV * rho * ni)/beta - (ni/beta)^2 - (sigmaV^2)/2) * t + ((sigmaV * rho
* ni)/(beta^2) + ((ni^2)/(beta^3))/2) * exp(- beta * Tmat) * (exp(beta * t) - 1) +
        (r0/beta - alpha/(beta^2) + (ni^2/beta^3)) * (1 - exp(- beta * t)) - ((ni^2/beta^3))/2 *
exp(- beta * Tmat) * (1 - exp( - beta * t)))
}

# equation (39)

a <- function(X, rho, sigmaV, alpha, beta, ni, i, Tmat, n, r0){
  return((- log(X) - M(rho, sigmaV, alpha, beta, ni, i * Tmat/n, Tmat, r0))/
         sqrt(S(rho, sigmaV, alpha, beta, ni, i * Tmat/n)))
}

# equation (40)

b <- function(X, rho, sigmaV, alpha, beta, ni, i, j, Tmat, n, r0){
  return((M(rho, sigmaV, alpha, beta, ni, j * Tmat/n, Tmat, r0) - M(rho, sigmaV, alpha, beta,
ni, i * Tmat/n, Tmat, r0))/
         sqrt(S(rho, sigmaV, alpha, beta, ni, i * Tmat/n) - S(rho, sigmaV, alpha, beta, ni, j *
Tmat/n)))
}

# equations (36)-(38). Returns final result for the LS model

LSprobApprox <- function(beta, theta, ni, Fs, Vs, sigmaV, rho, r0, Tmat, n){
  # beta..Vasicek parameter (intensity)
  # theta .. Vasicek parameter (long-term)
  # ni .. Vasicek vol
  # Fs .. barrier
  # Vs .. assets
  # sigmaV ... asset vols
  # rho .. correlation
  # r0 ... interest rate at time s
  # Tmat ... maturity
  # n ... number of discretized intervals of time T

# check for special cases
  if(any(is.na(c(beta, theta, ni, Fs, Vs, sigmaV, rho, r0, Tmat, n)))){
    return(NA)
  }else if(Fs > Vs){
    return(1)
  }else{

    X <- Vs/Fs
    alpha <- beta * theta

    q <- pnorm(a(X, rho, sigmaV, alpha, beta, ni, 1:n, Tmat, n, r0))
```

```
  for(i in 2:n){

    q[i] <- q[i] - sum( pnorm(b(X, rho, sigmaV, alpha, beta, ni, i, 1:(i-1), Tmat, n, r0)) *
q[1:(i-1)] )

  }

  return(sum(q))
 }
}
```

Appendix 6: Implying CDS probabilities of default – R code implementation

```
# CDSval function valuates the CDS given the selected PD. It implements equations (65)
and (67)

CDSval <- function(piCDS, parms){
 n <- parms[1]
 Tmat <- parms[2]
 r <- parms[3]
 s <- parms[4]/10000
 delta <- parms[5]

 # consecutive numbers of cash flows
 i <- 1:(n * Tmat)
 # payment periods
 t_i <- 1/n * i

 sellerPV <- sum((1-piCDS)^i * exp(-r * t_i) * s/n +
                 (1-piCDS)^(i - 1) * piCDS * exp(- r * (t_i - 0.5/n)) * s /(2 * n))
 buyerPV <- sum((1-piCDS)^(i - 1) * piCDS *
                (1 - delta) * exp(- r * (t_i - 0.5/n)))
 return(abs(buyerPV - sellerPV))
}

# CDSpds finds implied PD for a given CDS

CDSpds <- function(n, Tmat, r, s, delta){
 # n ... number of payments per year
 # Tmat ... time to maturity
 # r ... risk-free interest rate
 # s ... CDS spread
 # delta ... recovery rate

 if(any(is.na(c(n, Tmat, r, s, delta)))){
  return(NA)
 }else{
  parameters <- c(n, Tmat, r, s, delta)
```

```
        ## searching for a minimum (root) of function CDSval
   periodPD <- optimize(f = CDSval, parms = parameters, maximum = FALSE, lower = 0,
upper = 1, tol = 10^-50)$minimum
        ## transforming quarterly PD into annualized PD
   annualPD <- 1 - (1 - periodPD)^n
   return(annualPD)
  }

}
```

Appendix 7: Vasicek parameters' calibration – R code implementation

```
library(optimx)
# Vasicek yield formula given Vasicek parameters

yieldVasicek <- function(tenors, a, b, sigma, r0){
  # a ... Vasicek intensity
  # b ... long-term average
  # sigma … volatility
  B <- (1-exp(- a * tenors))/a
  A <- (b-sigma^2/(2*a^2)) * (B - tenors) - (sigma^2 * B^2)/(4*a)

  y <- 1/(tenors) * (B * r0 - A)
  return(y)
}

# returns sum of squares for given Vasicek parameters

estimating <- function(parameters, tenors, yields){

  a <- parameters[1]
  b <- parameters[2]
  sigma <- parameters[3]
# 3rd element presents 1-year Treasury yield which I assumed as a risk-free rate
  r0 <- yields[3]

  y <- yieldVasicek(tenors, a, b, sigma, r0)
  differences <- yields - y
  # - in order to maximize sum of squares
  return(sum(differences^2))
}

# returns Vasicek parameters for given market yield

vasicekEstimation <- function(yield, vols, tenors){
# yield … market yields
# vols … historical volatility of the risk-free rate
# tenors .. tenors corresponding to vector yield
```

```
  yield <- as.numeric(yield)
  vols <- as.numeric(vols)

  if(any(is.na(yield))){
    indexValid <- which(!is.na(yield))
  }else{
    indexValid <- 1:length(yield)
  }

# skip N/A values
  yield <- yield[indexValid]
  tenors <- tenors[indexValid]

  # limits
  upperVol <- 2 * vols
  lowerVol <- 0.5 * vols
  upperB <- 0.1
  upperA <- 5

  estimation <- optimx(par = c(0.01, mean(yield), vols), fn = estimating, lower = c(0,-Inf,
lowerVol), upper = c(upperA, upperB, upperVol), method = "nlminb", tenors = tenors,
yields = yield)

# vector of the three estimated parameters and value of accuracy of a fit
  estimates <- c(estimation$p1, estimation$p2, estimation$p3, estimation$value)
  return(estimates)
}
```

Appendix 8: Monte Carlo simulations in Black-Cox framework – R code implementation

```
# monteCarloV_t function generates part for V_t given the parameters

monteCarloV_t <- function(n, mu, V_0, sigma_V, numOfSim){
  normal <- rnorm(n-1)
  period <- 1/(n-1)
  rV_t <- c(0, normal * sigma_V * sqrt(period) + (mu - sigma_V^2/2) * period)

  # you can sum up log-returns and obtain log-returns of longer period
  V_t <- exp(cumsum(rV_t)) * V_0
  return(V_t)
}

# monteCarloProbs returns PD estimate given parameters and number of simulations

monteCarloProbs <- function(V_0, sigma_V, Ft, N, rt, timeToMaturities){
  # V_0 ... value of assets
  # sigma_V ... volatility of assets
  # Ft .. values of debt
```

```
  # N .. number of simulations
  # rt .. risk-free rate
  # timeToMaturities ... time discretization vector

  # in case of any NAs
  if(any(is.na(c(V_0, sigma_V, Ft, N, rt, timeToMaturities)))){
    return(NA)
  }

  Kt <- Ft * exp(-rt * timeToMaturities)

  ## if we already start below the threshold in Black-Cox world
  if(Kt[1] >= V_0){
    return(1)
  }

  n <- length(Kt)
  vectorOfDefaults <- numeric(N)
  for(i in 1:N){
    vectorOfDefaults[i] <- any(Kt>=monteCarloV_t(n, rt, V_0, sigma_V, i))
  }

  return(mean(vectorOfDefaults))
}
```

Appendix 9: Cox Proportional Hazard Regression – R code implementation

```
  coxfunction <- function(indicesTaken, data){
    # data .. data frame object consisting of columns: days to default data, censorship data
and a list of columns with independent variables.
    # indicesTaken .. a vector of numbers that lists consecutive numbers of columns which
account for independent variables
    # column 2 of data determines days to default or to censorship
    # column 3 of data binary (0/1) determines censorship

    ## creates a formula object
    fmla <- as.formula(paste("Surv(data[, 2], data[, 3]) ~ ",
paste(colnames(data)[indicesTaken], collapse= " + "), "+ cluster(data[, 1])"))
    CoxHazard <- coxph(formula = fmla, data=data)
    return(CoxHazard)
  }
```

Appendix 10: List of all analyzed companies

The table below lists Bloomberg tickers for each company that has been analyzed in this study.

Table 15: List of all analyzed firms

BGG	GLW	NYT	GCO	CVX	**WNDXQ**	APD	**NRTLQ**
PH	CMI	JCP	MDP	XOM	CL	CCK	OKE
BA	DE	PHM	**FLTWQ**	RDC	CAG	AVY	PCG
CAT	DOV	HD	**FBNIQ**	SLB	GIS	BLL	EXC
PBI	ETN	MCD	TGT	HES	IFF	WY	PEG
GE	PKI	LUB	KBH	MRO	K	BMS	EIX
MMM	EMR	WMT	F	HAL	KR	WOR	SO
RTN	**UAWGQ**	**MTLQQ**	**KMRTQ**	HP	VZ	NC	AEP
R	FDX	SNA	TGNA	ASH	T	NEU	DTE
TKR	**GWOW**	SWK	GPS	MDR	JNJ	NL	WMB
UTX	USG	CAL	GPC	HSY	MRK	DOW	XEL
NAV	**HRZIQ**	**EKDKQ**	GT	PEP	PFE	ECL	ED
SHW	GD	CMCSA	FL	KO	ABT	FMC	D
HON	GWW	TJX	NKE	PG	BCR	WLB	DUK
CVGYQ	**UALAQ**	**RHDCQ**	**PRDCQ**	MO	BAX	**BHMSQ**	ETR
TXT	CR	COST	DDS	SVU	BDX	KMB	NEE
JCI	DLX	**ZETHQ**	**ACKH**	SYY	BMY	NEM	**ENRNQ**
UNP	ITW	BSET	LB	ADM	HUM	HPQ	CNP
PCAR	IR	VFC	LOW	AVP	LLY	IBM	
HRB	**OWENQ**	**DCNAQ**	BC	**GAPTQ**	MDT	INTC	
XRX	MAS	SPXC	SPGI	RAD	AA	AAPL	
ITT	**AAMRQ**	WHR	CVS	**FLMIQ**	DD	ADSK	
MZIAQ	NSC	**HTMXQ**	**CCTYQ**	CPB	PPG	**8384662Q**	
DALRQ	NOC	**RSHCQ**	NWL	CLX	LPX	HRS	
SCI	SKY	JWN	OXY	TAP	IP	MSI	

This table lists Bloomberg tickers for each analyzed company. The emboldened 29 companies in the above table are those that defaulted between February 1990 and August 2017 whereas the rest were still active as of August 4, 2017.

Source: own work.