UNIVERSITY OF LJUBLJANA
SCHOOL OF ECONOMICS AND BUSINESS

MASTER’S THESIS

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# EFFICIENT PORTFOLIO DIVERSIFICATION OF THE EUROPEAN LARGE AND SMALL-CAP STOCKS

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LUKA VLASTELICA

## AUTHORSHIP STATEMENT

The undersigned LUKA VLASTELICA, a student at the University of Ljubljana, School of Economics and Business, (hereafter: SEB LU), author of this written final work of studies with the title EFFICIENT PORTFOLIO DIVERSIFICATION OF THE EUROPEAN LARGE AND SMALL-CAP STOCKS, prepared under supervision of izr. prof. dr. Igor Lončarski

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sl. – Slovene

**CAL** – (sl. premica razdelitve kapitala); capital allocation line

**CAPM** – (sl. model vrednotenja dolgoročnih naložb); capital asset pricing model

**CML** – (sl. premica kapitalskega trga); capital market line

**ERC portfolio** – (sl. portfelj ponderiran z enakomernimi prispevki k tveganju); equal risk contributions portfolio

**EW portfolio** – (sl. portfelj z enakimi utežmi); equally weighted portfolio

**FW portfolio** – (sl. portfelj ponderiran z vrednostmi knjigovodskih postavk podjetij); fundamentally weighted portfolio

**MCW portfolio** – (sl. tržno ponderiran portfelj); market-cap weighted portfolio

**MD portfolio** – (sl. najbolj razpršen portfelj); most diversified portfolio

**MSR portfolio** – (sl. portfelj z najvišjim Sharpovim kazalnikom); maximum Sharpe ratio portfolio

**MV portfolio** – (sl. portfelj z najnižjo varianco); minimum variance portfolio

**PCA** – (sl. analiza glavnih komponent); principal component analysis

**SML** – (sl. premica vrednosti naložb); security market line

# INTRODUCTION

In the recent two decades when index investing became very popular, academic researchers and investment professionals started questioning the efficiency of the market-cap weighted indexes. As a result, a number of different weighting approaches have been proposed which vary by their ultimate goal. Some approaches aim to maximize return, some focus on minimizing variance or extreme risk, while other try to maximize risk-adjusted return.

In 2017, assets under management in passive funds represented around 20% of all investments, whereas decade ago passive funds represented only 8% (Sushko & Turner, 2018, p. 114). Although the passive investing is still much smaller than active investing, in the last 10 years the cumulative fund flows to the former were around 3 trillion USD while the latter experienced roughly the same outflows (Sushko & Turner, 2018, p. 114-115). While people started investing heavily into index funds, majority of investments went to market-cap weighted indexes such as the S&P 500 (US stock market index), FTSE 100 (UK stock market index), DAX (German stock market index) etc. But researchers such as Haugen and Baker (1991), Grinold (1992) showed long-ago that cap-weighted stock indexes are suboptimal investments, as it is possible to construct portfolios with the same or even higher expected return and lower variance. The market-cap weighted portfolios are efficient only under the very limiting theoretical assumptions, which are: all investors have the same expectations about risk and expected return for all securities, short selling is available to all investors, the absence of all taxes and limitation of investment opportunity set to only securities in the market-cap weighted index.

As a consequence of the inefficiency of the market-cap weighted indexes and the increased investor's preference for passive investing, researchers started proposing different approaches to efficiently diversify portfolios. Arnott, Hsu and Moore (2005) proposed to weight stocks by their fundamentals such as book value, cash-flow, revenues, sales, dividends and total employment. DeMiguel, Garlappi and Uppal (2009) showed that equally weighted portfolio strategy can offer favourable results. Clarke, De Silva and Thorley (2006) found that minimum variance portfolios have higher annualized means and lower standard deviations resulting in much higher Sharpe ratios compared to a market-cap weighted portfolio. Another method for portfolio construction is maximum diversification, which was proposed by Choueifaty and Coignard (2008). Maillard, Roncalli and Teiletche (2010) proposed equally-weighted risk contributions portfolios, where the risk contribution from each portfolio component is made equal. Martellini (2008), Amenc, Goltz, Martellini and Retkowsky (2010) proposed a novel approach for constructing an efficient index which aims at maximizing the Sharpe ratio.

The purpose is to investigate different approaches to portfolio diversification, i.e. the market-cap weighting, the fundamentals weighting, the equal weighting, the minimum variance portfolio, the maximum diversification portfolio, the equal risk contribution portfolio and the maximum Sharpe ratio portfolio. I want to highlight the advantages and disadvantages of different approaches, show their similarities and differences and contribute to their better understanding. I will analyse how different approaches would have performed in the past using the European large-cap stocks and small-cap stocks data. To my knowledge the analysis on the samples of small and large-cap stocks is new and has not been previously researched. Additionally, I will construct the optimized portfolios with differently estimated stock return moments to see if advanced statistical and econometric methods can help enhance the out of sample portfolio performance. Therefore, the thesis will provide new insight on how different portfolio weighting approaches behave on small and large-cap stock portfolios.

The objective is to review the financial theory and study the approaches to portfolio diversification based on the relevant scientific literature. Furthermore, I will use statistical and quantitative research methods to construct portfolios with different weights and test which approach to portfolio diversification has the highest out-of-sample Sharpe ratio. Because portfolio managers have different objectives and constraints, I will also look at the more specific portfolio evaluation statistics such as the tracking error versus the cap-weighted benchmark, information ratio and different measures of extreme risk and concentration. The research will be performed on the European stock market.

The research hypotheses, that were set based on the reviewed literature, are:

1. The market-cap weighted portfolio is inefficient as one can construct portfolio with higher out-of-sample Sharpe ratio.
2. The equally-weighted portfolio has a higher return and a higher Sharpe ratio but also a higher risk than the market-cap weighted portfolio.
3. The fundamentally-weighted portfolio has a higher Sharpe ratio than the market-cap weighted portfolio.
4. The minimum variance portfolios with improved estimates of covariance matrix have the lowest standard deviations.
5. The maximum Sharpe ratio portfolios with improved estimates of the stock return moments have the highest Sharpe ratios.
6. There are no differences if the approaches are applied to the large or small-cap stocks.

The structure of the master's thesis is as follows. The first part of the thesis discusses the portfolio theory, i.e. the portfolio risk and return, modern portfolio theory and capital asset pricing model (hereinafter: CAPM). In the second part, I describe different approaches to portfolio diversification and review previous findings. I include the market-cap weighting, the fundamentals weighting, the equal weighting, the minimum variance portfolio, the equally weighted risk contributions portfolio, the most diversified portfolio and the maximum Sharpe ratio portfolio. In the third part of the thesis, I take a brief look at the estimation of portfolio moments, namely, the estimation of expected returns, covariance matrix, coskewness matrix and cokurtosis matrix. In the last part, I combine the knowledge from the first three parts to create simple and optimized portfolios according to different approaches. The final part concludes on the research findings of the master's thesis.

# 1 PORTFOLIO THEORY

In the first part of the thesis, I cover financial theory that lays the foundations for the future research. First, I go through the basic concepts of a portfolio risk and return, explain a covariance between stocks and discuss basic logic for portfolio diversification. Then I present Markowitz's modern portfolio theory, which is followed by the CAPM. I explain why the CAPM was widely criticised and present the factor models that were developed as a consequence.

## 1.1 Portfolio risk and return

In this part, I show how to analyse portfolio of two risky assets as this can be then easily extended to portfolio of many assets. Simple measures are portfolio expected return, variance or standard deviation and covariance. Additionally, I show the benefits of simple diversification and how many securities are sufficient to a diversify portfolio.

### 1.1.1 Portfolio expected return

Portfolio expected return is calculated as the weighted average of stocks expected returns and their respective weights in a portfolio. This can be written as:

$$E\left(r_p\right) = w_1 * E(r_1) + w_2 * E(r_2) \tag{1}$$

Where $w_1$ and $w_2$ are security weights and $E(r_1)$ and $E(r_2)$ are expected returns.

In the matrix notation for the case of many securities the portfolio expected return can be written as:

$$w^T * \mu \tag{2}$$

Where $w^T$ is a transposed vector of weights and $\mu$ is a vector of stocks expected returns.

### 1.1.2 Portfolio variance and covariance

Variance of an individual stock is a measure of risk and it measures how dispersed are data points around their mean value (Investopedia, n.d.). In contrast to the portfolio expected return, portfolio variance is not a weighted average of individual variances but it depends on the variance of component securities and also the covariance between them. Covariance is simply a measure of how returns of securities move together (Bodie, Kane & Marcus, 2011, p. 199). Thus, one can write the variance of a two-stock portfolio as follows:

$$\sigma_p^2 = w_1^2 * \sigma_1^2 + w_2^2 * \sigma_2^2 + 2 * w_1 * w_2 * Cov(r_1, r_2) \quad (3)$$

Where $w_1$ and $w_2$ represent security weights, $\sigma_1^2$ and $\sigma_2^2$ are security variances and $Cov(r_1, r_2)$ is covariance between security returns.

Standard deviation is also a measure of the dispersion of a data and is calculated as the square root of variance. Because the variance is squared and standard deviation is not, one use the latter to describe annual volatility of stocks. Standard deviation is expressed in percentages, the same as the expected return, whereas variance does not have any units.

Other important concepts for portfolio diversification are covariance and correlation. As already mentioned, covariance measures the degree to which two securities move together. It can take value from $-\infty$ to $+\infty$ and it can be computed from a correlation coefficient. To get the sense how much two stocks are correlated, it is better to look at the correlation coefficient as it can range from -1 to +1, where -1 means that two stocks are perfectly negatively correlated, +1 represents perfect positive correlation and 0 means that two stocks are not correlated at all (Bodie, Kane & Marcus, 2011, p. 201). One can calculate the covariance from the correlation coefficient as:

$$Cov(r_1, r_2) = \rho_{1,2} * \sigma_1 * \sigma_2 \quad (4)$$

Where $\rho_{1,2}$ is correlation coefficient between two stocks and $\sigma$ is standard deviation of each stock.

The equation (4) can then be inserted into equation (3) and the result is:

$$\sigma_p^2 = w_1^2 * \sigma_1^2 + w_2^2 * \sigma_2^2 + 2 * w_1 * w_2 * \sigma_1 * \sigma_2 * \rho_{1,2} \quad (5)$$

This means that portfolio variance does not depend only on the individual variances of stocks but also on the correlation between stock returns. In the extreme case when stocks are perfectly positively correlated, the portfolio standard deviation will be the weighted average of the individual standard deviations. Thus, in order to diversify a portfolio and reduce the overall risk, investors are motivated to add stocks that have low or even negative correlation. The second extreme case would be perfect negative correlation. This would mean that investors can choose the portfolio weights that would result in a portfolio with standard deviation equal to zero (Bodie, Kane & Marcus, 2011, p. 201).

In a generalized form for many securities, I calculate variance-covariance matrix, which has the $n*n$ dimensions. The diagonal elements in the matrix are stock variances and elements below and above the diagonal are stock covariances, where the covariance in the i-th row and the j-th column is the same as the one in the j-th row and the i-th column. This makes covariance matrix symmetric. In the matrix notation, portfolio variance is calculated as:

$$w^T * \sum * w \quad (6)$$

Where w is a vector of weights, $w^T$ is a transposed vector of weights and $\sum$ is the covariance matrix.

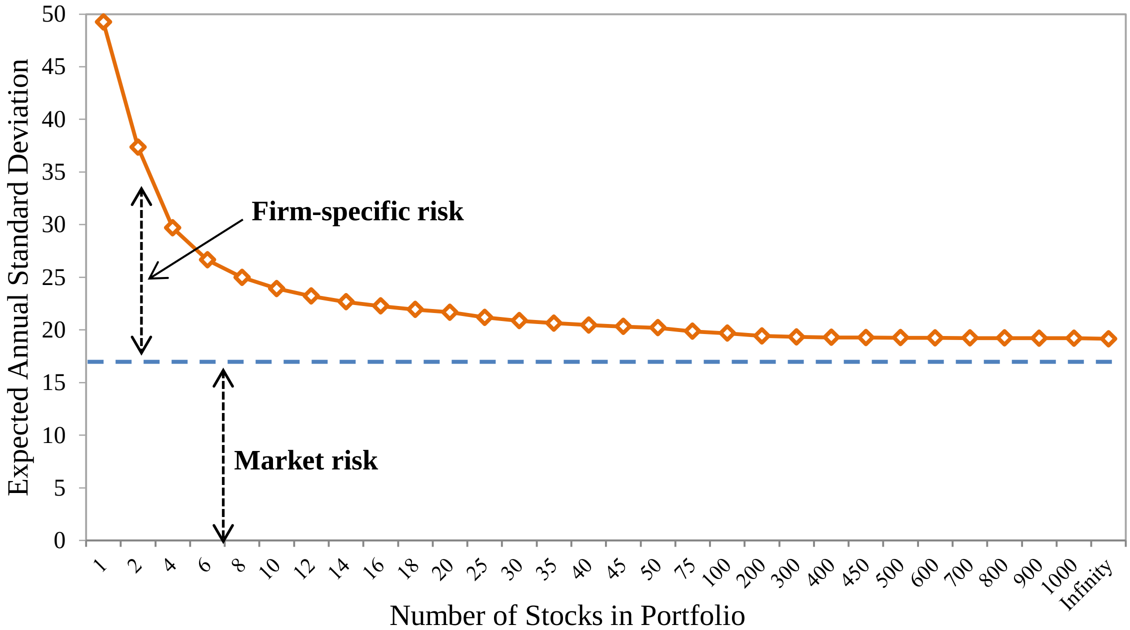
### 1.1.3 Risk and portfolio diversification

As demonstrated, investors are motivated to add stocks with low correlation in returns to their portfolio to reduce risk. Reduction in portfolio risk is possible because there are two different sources of uncertainty. The first one is macroeconomic risk, which includes changes in interest rates, inflation, exchange rates and business cycle. This is the risk that affects all companies in the economy. The second source of risk is firm-specific risk and it includes factors that affect a particular company such as personnel changes, efficiency of research and development, raw materials costs etc. These factors affect the particular company but do not have much effect on other companies in the economy. The risk that cannot be diversified away is called the market risk, or non-diversifiable risk, or systematic risk. The risk that can be diversified is called the firm-specific risk, or unique risk, or diversifiable risk or non-systematic risk (Bodie, Kane & Marcus, 2011, p.201).

The benefits of diversification are bigger if one includes stocks with low correlation in returns. This means that portfolio risk is reduced more if one adds stocks from unrelated industries, e.g. if I hold stock of one oil company and then diversify portfolio with another oil company, the effect of diversification will be small because they are both massively exposed to oil prices. Hence, a better choice would be to add stock from health care industry, utilities, technology or any other unrelated industry. Another possibility to diversify portfolio is an inclusion of international company stocks because they are not exposed to the same market risk. In the past, the benefit of diversifying portfolio to international stocks was huge but with globalization and increase in the world trade firms became exposed to more common risk factors.

However, the benefits of diversification can be reaped quickly as firm-specific risk declines very fast at the beginning. This can be seen in Figure 1 which shows how risk, measured as the expected annual standard deviation of portfolio returns, declines with the inclusion of more stocks into portfolio. Researchers are not in agreement about how many stocks should be included in an equally diversified portfolio as some suggest that 10-15 stocks are sufficient to get rid of the firm-specific risk (Evans & Archer, 1968), while others recommend holding between 30-40 stocks (Statman, 1987). Tang (2004) came to the conclusion that a portfolio of 20 stocks is sufficient to eliminate 95% of the firm-specific risk.

Figure 1: Portfolio diversification and firm-specific risk




*Source: Statman (1987, p. 355).*

## 1.2 Modern portfolio theory

Modern portfolio theory is presented next and it shows how correlation between stocks can be exploited to create portfolios that have minimized variance at each level of target return. Firstly, I take a look at the example of two risky securities and then present the Markowitz's modern portfolio theory on the case of many risky securities.

### 1.2.1 The case of two risky securities

When initially looking at the example of two risky securities, one can see why correlation between stocks really matters. An example, I invest into two stocks: one technology stock with high expected return and high standard deviation and one consumer staples stock with low expected return and low standard deviation. Expected return and standard deviation for two stocks are presented in Table 1.

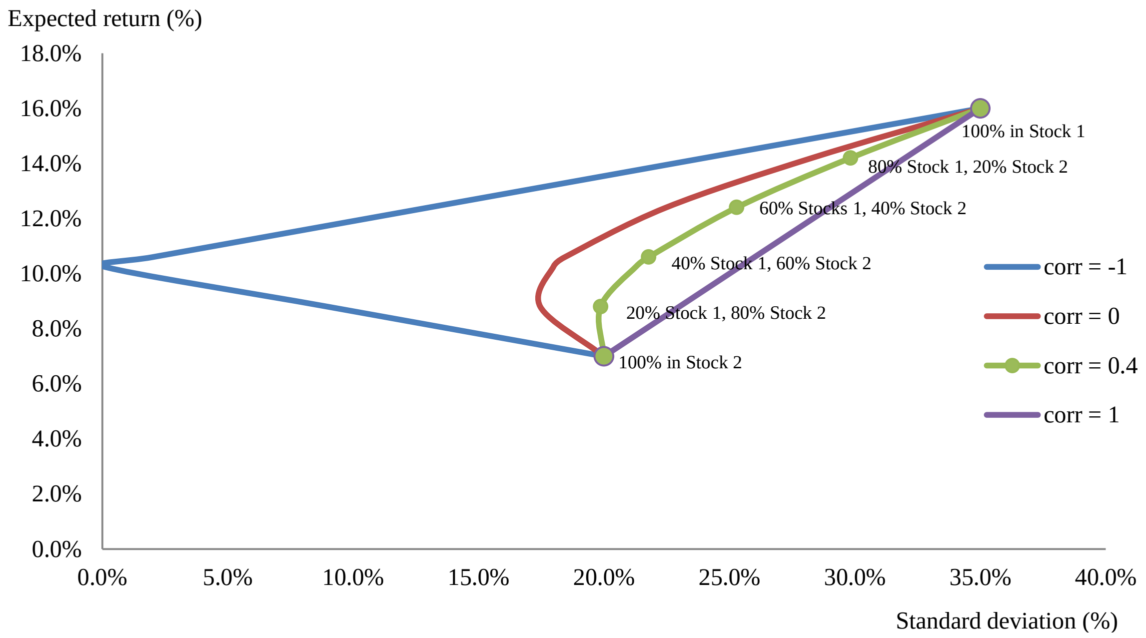
Table 1: Two stock portfolio

| | Expected return | Standard deviation |
|---|---|---|
| Stock 1 – Technology stock | 16% | 35% |
| Stock 2 – Consumer staples stock | 7% | 20% |

*Source: Own work.*

If I invest into these two stocks, portfolio expected return will vary from 16% (100% invested into stock 1) to 7% (100% invested into stock 2). As already known, portfolio standard deviation will change differently and it will depend on a correlation between stocks. In the Figure 2, one can see how diversification effect is bigger if correlation is lower. As the correlation between stocks gets lower, the portfolio's standard deviation can be reduced more. In the first extreme case when correlation is equal to 1, there will be no benefits to diversification and in the second extreme case when the correlation is equal to -1, I would be able to perfectly hedge a portfolio of two stocks. Real world correlations between stocks will usually be higher than 0.5 and negative correlations are very difficult to find.

Figure 2: Correlation between two stocks and portfolio standard deviation




*Source: Own work.*

Additionally, one can see in Figure 2 how investor who prefers low risk investments and is 100% invested in Stock 2 can increase his expected return and reduce standard deviation. When correlation between investments is 0.4, he can sell 20% of investment in stock 2 and buy 20% of Stock 1. This will increase his expected return to 8.8% and slightly decrease standard deviation to 19.9%. But this process of increasing expected portfolio return and decreasing risk cannot go indefinitely. The question is which portfolio weights maximize investors risk adjusted return? I answer that question next.

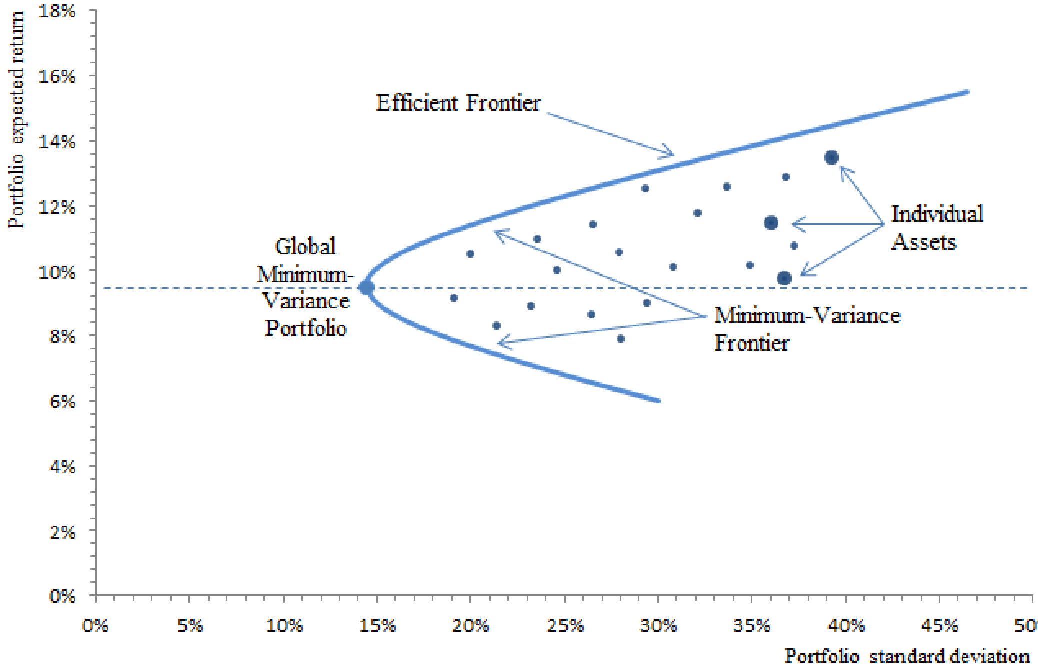
### 1.2.2 The case of many risky securities

Now, that I know how diversification works, I can extend the portfolio construction problem to the case of many risky securities. Markowitz (1952) was the first who showed how portfolio selection can be done in practice. He showed how investors behave and gave guidance for portfolio construction. The expected return-variance framework that he presented is used to identify the mean-variance efficient combinations.

Markowitz's mean-variance analysis simply means that based on the stock input data for expected returns, variances and covariances, one can calculate portfolios with minimum-variance for each targeted expected return. The result is minimum-variance frontier as shown in the Figure 3. All the individual assets (stocks) lie on the right side of the minimum-variance frontier, which means that diversified portfolios are better than individual stocks. But this is true only if short sales are allowed, otherwise it is possible that some stocks will lie on the frontier, e.g. the stock with the highest expected return will be on the frontier because this is the only way to achieve that high return and, thus, it is also the one with the minimum-variance. However, diversification of portfolios leads to higher expected returns and/or lower standard deviations (Bodie, Kane & Marcus, 2011, p. 211; Cuthbertson & Nitzsche, 2004, p.124–126).

In the Figure 3, one can see that portfolio with the lowest standard deviation is called the global minimum-variance portfolio. The portfolios that lie on the minimum-variance frontier below the global minimum-variance portfolio are inefficient because there are always portfolios with the same variance but higher expected return that lie directly above them. Hence, the portfolios that lie above the global minimum-variance portfolio are called efficient frontier of risky assets (Bodie, Kane & Marcus, 2011, p. 211; Cuthbertson & Nitzsche, 2004, p.124–126).

Figure 3: The minimum-variance frontier of risky assets




*Source: Bodie, Kane & Marcus (2011, p. 211).*

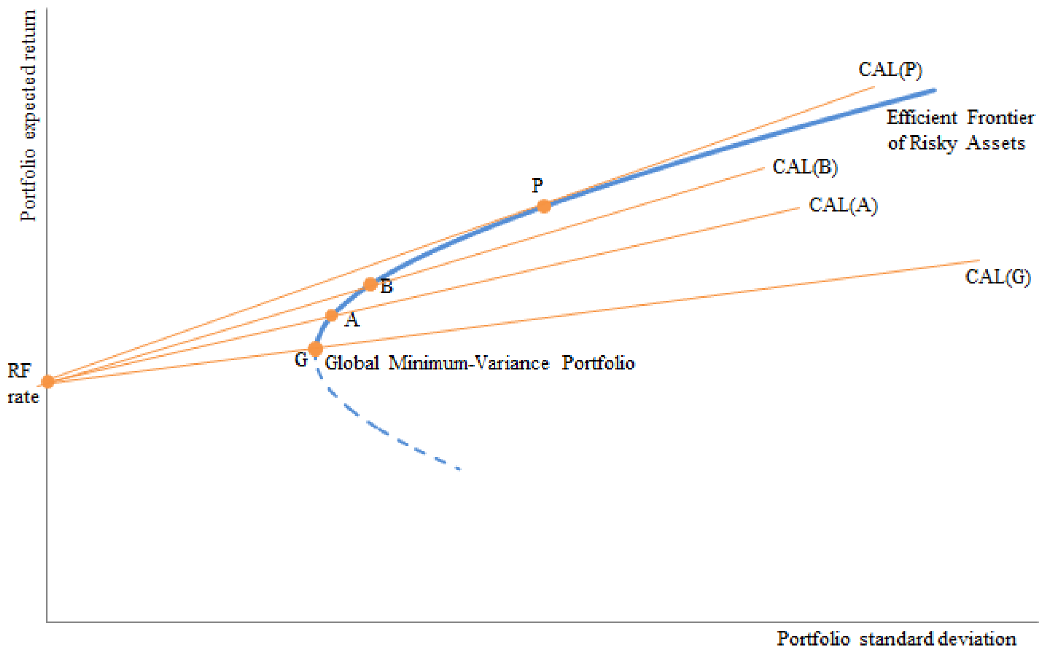
The estimation of an efficient frontier is done in six steps (Cuthbertson & Nitzsche, 2004, p.125–126):

1. Investor chooses an arbitrary target return on the portfolio (e.g. 10%).

2. He arbitrarily chooses the proportions (weights) to invest in each stock such that target portfolio return is achieved. Portfolio weights must sum to 1.
3. He calculates variance (standard deviation).
4. He repeats the process under steps (2) and (3) until he gets the weights that minimize portfolio variance (standard deviation).
5. When he gets the portfolio weights that satisfy target return criterion and have minimum standard deviation, he has the weights that present the first point of the efficient frontier.
6. Lastly, he chooses another arbitrary target return on the portfolio (e.g. 9%) and repeats the process above. He does this until he gets the portfolio weights that minimize standard deviation. Then he repeats the process until the points connect into efficient frontier.

After having calculated efficient frontier, one must figure out which of these portfolios are the best choice for our investor. To do this, I introduce a risk-free asset into our analysis. When introducing the risk-free rate into the analysis, it allows one to search for the steepest capital allocation line (hereinafter: CAL). This is the one that is tangent to the efficient frontier and it has the highest reward-to-volatility (Sharpe) ratio. The optimal risky portfolio is in point P, where CAL(P) is tangent to the efficient frontier (Bodie, Kane & Marcus, 2011, p. 211). The second step of the mean-variance analysis is shown in the Figure 4.

Figure 4: Efficient frontier and capital allocation lines




*Source: Bodie, Kane & Marcus (2011, p. 215).*

The Sharpe ratio which measures the reward-to-volatility or risk adjusted return is calculated as the excess return of a portfolio over a risk-free rate divided with the portfolio standard deviation. This can be written as follows:

$$Sharpe\ ratio_P = \frac{E(r_P) - r_f}{\sigma_P} \quad (7)$$

Therefore, the highest Sharpe ratio will be when CAL is the steepest. In that point, a reward to volatility is the highest. The implication is that none of the investors will be interested in holding a portfolio in point A, B or G. The portfolio P which has the highest Sharpe ratio will thus be held by every investor and is called the optimal risky portfolio. This is where the work of a portfolio manager ends as he will offer the same optimal risky portfolio to every client if there are no additional portfolio constraints required by the client (Bodie, Kane & Marcus, 2011, p. 214–215).

The last question is how much will client invest in a risky portfolio and how much in riskless securities. This is called a separation principle and it was first described by Tobin (1958). This decision is independent of the portfolio manager's work and is based solely on the investor's risk preference. When looking at the Figure 4, investor who is reasonably risk averse will hold a combination of optimal risky portfolio and risk-free asset on the CAL(P) left of the point P. Depending on his risk aversion, he may choose to invest 40% into riskless securities (T-Bills in the US or German bonds in the Europe) and 60% into optimal risky portfolio. All the investors who are risk averse will hold a combination of riskless asset and risky portfolio on the CAL(P) left of the point P. In point P, investor will hold 100% of his savings in the optimal risky portfolio while right of the point P on CAL(P) investor will hold levered portfolio as he will borrow money to increase his position in the optimal risky portfolio (Cuthbertson & Nitzsche, 2004, p.126–132).

The mean-variance portfolio optimization process done by portfolio manager can be summed up into 2 main steps (Bodie, Kane & Marcus, 2011, p. 212–214):

1. In the first step, portfolio manager needs the input list which includes estimates of expected returns of each security and estimate of covariance matrix. This is the most difficult task in a portfolio optimization as bad estimates lead to poor optimization and poor performance of the portfolio. The portfolio manager needs $n$ estimates of expected returns and $n(n - 1)/2$ estimates of the covariance matrix to perform portfolio optimization. If the portfolio includes 50 securities, the input list will consist of 50 estimates of expected returns and 1,225 estimates of covariances. This is a very challenging task and I will review the whole process in detail in the third part of the thesis.
2. After I have estimated the input list, I can calculate the efficient frontier and the optimal risky portfolio with the highest Sharpe ratio. In practice this is done by feeding the input list into the optimization program. However, the optimal risky portfolio can differ for different clients because of additional constraints they require. Some institutional investors are prohibited from taking short positions, some require minimal expected dividend yield while others want to invest only in socially responsible companies.

Portfolio managers can tailor the portfolio to everyone's needs but this comes at the cost of lower reward-to-volatility ratio.

## 1.3 Capital asset pricing model

Markowitz portfolio theory showed how investors will optimize their portfolios while the capital asset pricing model presents how economic equilibrium in capital markets is formed. These two works present part one and part two of a microeconomics of capital markets (Markowitz, 1991, p. 469). Below, I present how equilibrium in capital markets is achieved under the CAPM assumptions and what are its implications.

### 1.3.1 CAPM and capital markets equilibrium

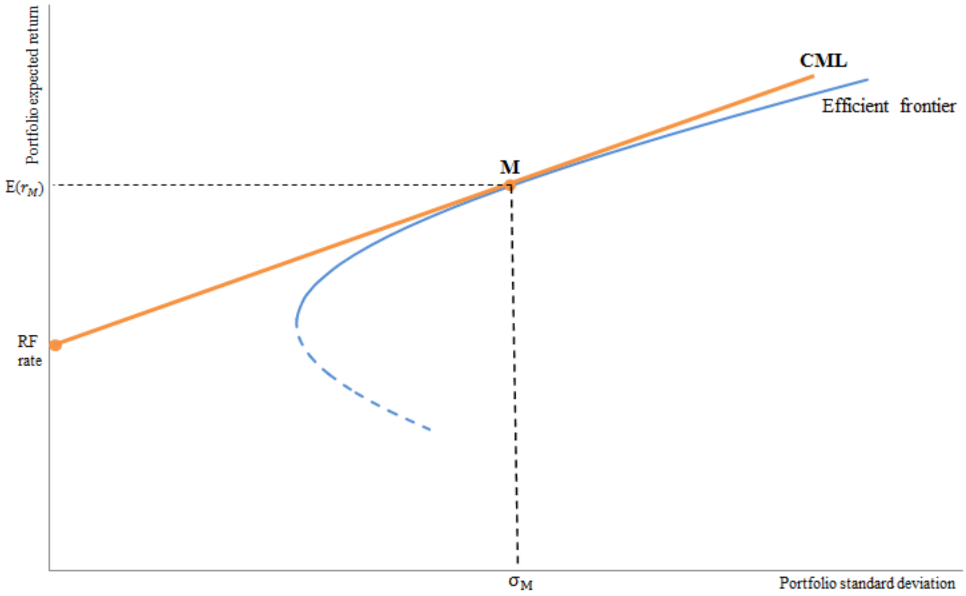
The capital asset pricing model was developed independently by Sharpe (1963; 1964), Lintner (1965) and Mossin (1966). The three authors started from Markowitz mean-variance analysis that describes investor behaviour and attempted to construct a model that would describe the capital market equilibrium of risky assets.

In order to be able to derive the CAPM the list of simplifying assumptions is needed (Bodie, Kane & Marcus, 2011, p. 281; Cuthbertson & Nitzsche, 2004, p.132–134):

1. Investors are price-takers. Their wealth is small in comparison with all the money in the market and they cannot influence the prices.
2. All investors aim to hold their portfolio for identical period and ignore what can happen afterwards.
3. Their investments are limited to the publicly traded financial assets (bonds, stocks and risk-free assets). They can lend and borrow money at the same risk-free rate and without limit.
4. There are no transaction costs (commissions and fees) and investors do not have to pay any taxes.
5. All investors behave according to the Markowitz mean-variance optimization model.
6. All investors behave in the same way, i.e. they have the same economic view of the world and analyse securities in the same way. This results in the same input list (expected returns and covariance matrix) used to calculate efficient frontier and optimal risky portfolio. This assumption is referred to as homogeneous expectations.

The implication of investors sharing the same beliefs and the absence of any market frictions is that they will behave in the same way. The mean-variance optimization will thus produce the optimal weights that are the same for all investors. This means the tangency portfolio on the steepest CAL is the same for every investor. Consequently, tangency portfolio (portfolio with the highest Sharpe ratio) will also be the market portfolio and CAL is now called the capital market line (hereinafter: CML). The capital market line goes from risk-free asset through market portfolio M. The weight of each stock in the market portfolio equals the market capitalization of that stock (stock price multiplied by number of shares outstanding) divided by the market capitalization of all stocks. The only difference among investors is their risk aversion that determines the amount of money invested in a market portfolio and a risk-free asset (Bodie, Kane & Marcus, 2011, p. 281–282; Cuthbertson & Nitzsche, 2004, p.132–134). The CML and the market portfolio are shown in Figure 5.

Figure 5: Capital market line and market portfolio




*Source: Bodie, Kane & Marcus (2011, p. 283) & Sharpe (1964, p. 432).*

One can see that each investor will hold market portfolio. Investors that are more risk averse will be on the left side of point M on the CML and will invest their money in market portfolio and lend the remainder at the risk-free rate. While the ones that are willing to take more risk, will borrow money at the risk-free rate and will invest in the market portfolio more than their initial wealth. They will position on the CML right of the point M (Sharpe, 1964, p. 432–433). When aggregating portfolios of all investors, lending and borrowing cancel out and value of all risky portfolios will equal the entire wealth of economy (Bodie, Kane & Marcus, 2011, p. 282).

If all that described above is true, then investors will not be motivated to do security analysis but will simply hold market portfolio. Thus, market index portfolio will be efficient and the investment process can then be broken down into two parts, creation of mutual funds by investment professional and individual decision of allocation to risky market portfolio and a risk-free asset (Bodie, Kane & Marcus, 2011, p. 283).

The above description can help us understand why passive investment strategies and market cap-weighted indexes became very popular in the recent decades. Even if the assumptions of the CAPM do not hold, the market cap-weighted indexes present a good starting point for portfolio diversification. Knowing that security analysis is difficult and costly, holding the market cap-weighted portfolio can actually prove to be an efficient strategy. I will discuss the implications of the CAPM for investment industry in the second chapter where I will also look at the different weighting schemes.

### 1.3.2 CAPM and the security market line

The CAPM does not only describe the equilibrium in capital markets but it also gives prediction about expected return of the asset based on its risk. The model uses insight that appropriate risk premium on an asset is determined by its contribution to the portfolio risk. That means investors are not concerned about stock's variance but its covariance with the market portfolio. It is the covariance with the market that contributes to the market portfolio risk (Bodie, Kane & Marcus, 2011, p. 285–287). This can be expressed this in equation (8) which represents expected return-beta relationship:

$$E(r_i) = r_f + \beta_i * [E(r_M) - r_f] \quad (8)$$

Where $E(r_i)$ is expected return of security i, $r_f$ denotes return on a risk-free asset, $E(r_m)$ is expected return of the market and $\beta_i$ (beta) measures the contribution of stock i to the variance of the market portfolio as a fraction of the total variance of the market portfolio (Bodie, Kane & Marcus, 2011, p. 287). β can be expressed as:

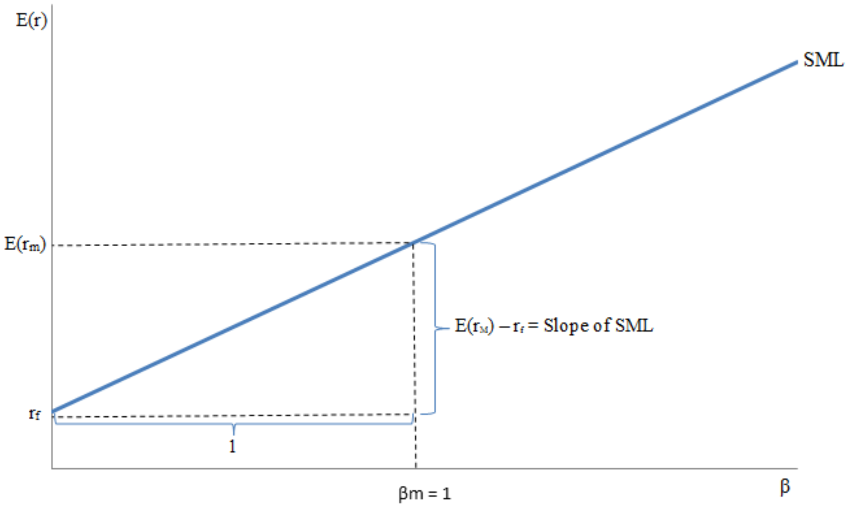
$$\beta_i = \frac{Cov(r_i, r_M)}{\sigma_M^2} \quad (9)$$

Where $Cov(r_i, r_m)$ is covariance of the i-th security with the market and $\sigma_m^2$ is variance of the market.

The expected return-beta relationship is in fact a reward-risk equation. Beta measures contribution of a stock to the variance of the market portfolio. Hence, the security's risk premium, as it can be seen in equation (8), is proportional to the beta and the market risk premium, i.e. $\beta_i * [E(r_m) - r_f]$. This relationship can be presented graphically as the security market line (hereinafter: SML). The relationship can be seen in Figure 6, where the slope of the SML is equal to the market risk premium. One can also see that market's beta equals 1, which can also be confirmed with equation (9) because covariance of the market with itself equals its variance. This implies that securities with beta lower than 1 will be less volatile than the market and will have lower expected return and vice versa (Bodie, Kane & Marcus, 2011, p. 289; Cuthbertson & Nitzsche, 2004, p.134).

When drawing the comparison of the CML and the SML, one can see that CML shows the risk premiums of efficient portfolios as a function of portfolio standard deviation. On the other hand, the SML shows risk premiums of individual asset as a function of asset risk (β) (Bodie, Kane & Marcus, 2011, p. 289). If the CAPM holds, then all securities should lie on the SML. When the securities lie on the SML, their expected returns compensate their risk. However, some securities can lie below the SML which means they have negative alpha and their return does not compensate for their risk. Those securities will be sold by investors so that their price would fall and expected return would rise to the point they would again be on the SML. The opposite is true for securities that lie above the SML, those securities have positive alpha and will be bought by investors until their price would rise and their expected return would fall to the point where they will again lie on the SML (Cuthbertson & Nitzsche, 2004, p.134–135).

Figure 6: The security market line




*Source: Bodie, Kane & Marcus (2011, p. 289).*

Thus, the value of the CAPM is not only to arrive to the equilibrium market portfolio but it can also be used to determine the fair expected return on the risky asset, or it can be used for capital budgeting decisions. It is useful because it provides the required rate of return on an investment, or a project while assuming that investors hold diversified portfolios and care only about systematic, or market risk (Bodie, Kane & Marcus, 2011, p. 290–292).

### 1.3.3 Criticism of the CAPM

In this subchapter, I want to briefly present the limitations of the CAPM as it was widely criticised because of its assumptions and simplicity. I do not aim to present all the problems and empirical research on the validity of the CAPM as there are too many and this is not the goal of this master's thesis.

First, I should mention the assumptions of the CAPM, some of them are unrealistic but they are necessary for the model to work as it is the case with the majority of models in economics and finance. One of the most famous critics is Roll's (1977) where he stated that fully diversified market portfolio is unobservable as it would have to include all types of assets including commodities, real estate, human capital etc. According to his critique, it is impossible to empirically test the CAPM. Therefore, to be able to test the model the equity indexes such as the S&P 500 are used as the proxies for market portfolio and the empirical tests are centred around the expected return-beta relationship (Bodie, Kane & Marcus, 2011, p. 298).

The CAPM fails these tests as a number of authors found that alpha values are not always zero. Black, Jensen and Scholes (1972) found that in the US market from 1931 through 1965 low beta stocks had positive alphas and high beta stocks had negative alphas. These findings were later confirmed by more authors what lead to the extensions of the CAPM. Some of the extensions are Intertemporal CAPM developed by Merton (1973) who extended analysis to multiple periods; Breeden (1979) proposed consumption CAPM where investors balance their current consumption and investments; Heaton and Lucas (2000) expanded the CAPM with labour income and entrepreneurial income. The attempts to extend the CAPM improved its predictive accuracy to some extent but none of the models solved all of the problems.

On the other hand, some authors showed that expected return-beta relationship still holds. Mayers (1973) presented single-period CAPM that included also non-marketable assets and showed that linear relationship between risk and expected return still holds. The difference that appears is that investors will hold different risk portfolios because of the inclusion of non-marketable assets such as human capital. Brennan (1970) came to the similar conclusion when he researched what happened when investors are in different tax brackets. He also found that risk-expected return relationship holds to some extent but investors will hold different optimal portfolios.

It is also not the case that all investors have the same information and homogenous expectations. In reality, portfolio managers can have different information and world views and will derive different optimal portfolios for their clients. Shefrin and Statman (2000) worked on behavioural portfolio theory which contradicts the idea of the CAPM and mean-variance optimization as they say that investors have more than one portfolio each aimed at certain goal – they are at the same time risk averse and risk seeking (they are buying both bonds and lottery tickets).

When it comes to the asset pricing, Fama and French (1993) added two additional factors to the CAPM that help predict average returns of stocks. The 3-factor model included market factor, small minus big (hereinafter: SMB) factor and high minus low (hereinafter: HML) factor. The SMB factor measures excess return of small cap stocks over big cap stocks and the HML factor measures higher returns of stocks with high book to market ratio. The model was later extended to four factors by Carhart (1997) with an inclusion of a momentum factor that describes the tendency of rising stocks to keep going up and declining stocks to keep going down. Fama and French (2015) later expanded their model to five factors as it has better explanatory power of stocks returns. Added factors are robust minus weak (RMW) and conservative minus aggressive (CMA), which measure the difference between returns on stocks with robust and weak profitability and the difference between returns on stocks with conservative and aggressive investment, respectively. However, even these factor models are not universal as Griffin (2002) found that country-specific factors better explain stock returns and have lower pricing errors than the world factor models. Foye, Mramor and Pahor (2013) showed that the 3-factor model derived by Fama and French (1993) performs poorly for the Eastern European countries and that its predictive accuracy is improved if the SMB factor is substituted with a term which proxies for accounting manipulation.

Despite the fact that CAPM is not perfect, it is still widely used in practice. The two main reasons for that are: its simplicity in decomposition of risk to systematic and firm-specific portion; and the fact that market portfolio may be itself efficient as many investment companies employ the best portfolio managers and professional analysts but there are very few that consistently beat the passive market index (Bodie, Kane & Marcus, 2011, p. 299).

# 2 APPROACHES TO PORTFOLIO DIVERSIFICATION

In this chapter, I will present different weighting approaches for portfolio diversification. The previous chapter showed that financial theory favours market capitalization weighting as every investor should hold the market portfolio which has the highest reward to risk ratio. But several authors found that capitalization weighted portfolios may be inefficient. Haugen and Baker (1991) showed that if certain assumptions are violated, i.e. if investors disagree about expected return and risk, if their investment returns are taxed, if short-selling is restricted, if investment in human capital is possible and if foreign investors are present, the market capitalization weighted portfolios will be suboptimal investments. They showed empirically for the US market between 1972 and 1989 that it was possible to construct portfolios with the same or higher expected return and considerably lower volatility. Grinold (1992) tested market cap-weighted benchmarks for 5 stock markets: The United States, Australia, the United Kingdom, Japan and Germany. He found that in the period between January 1973 and April 1991 all benchmarks except DAX for Germany were not efficient.

The academic literature today is clear that if even one of the CAPM assumptions does not hold, the market portfolio will be inefficient. Numerous researchers have demonstrated that market capitalization weighted indices are bad proxies of the market portfolio (Goltz & Le Sourd, 2011) and evidence on the inefficiency of the market-cap weighted portfolios lead researchers and industry professionals to propose alternative weighting approaches. The alternative approaches presented are equal weighting, fundamentals weighting, minimum variance portfolio, equally weighted risk contributions portfolio, the most diversified portfolio and maximum Sharpe ratio portfolio.

## 2.1 Market-cap weighted portfolio

The theoretical basis for market capitalization weighted portfolio (hereinafter: MCW portfolio) is the CAPM. Even though it was shown that capitalization weighted portfolios are inefficient, the majority of equity indices are market-cap weighted, such as the S&P 500, Russell 1000, FTSE 100, DAX and many others. A lot of individual and institutional investors like the idea that it is best to hold a proxy index for a market portfolio which resulted in an ever-growing investment in stock market indexes. However, this should not be a surprise because several authors such as Malkiel (1995) found that active investment management does not add value on average. In his study of the performance of mutual funds between 1971 and 1991, he found that the mean α was slightly negative but indistinguishable from zero which implicates that on average professional investment managers were unable to beat the passive strategy of investing in the market-cap weighted S&P 500 index. That result was also subject to survivorship bias because he took into account only mutual funds that survived through that period. The obtained results were also before fees which meant that after deducting management fees the average mutual fund underperformed the benchmark cap-weighted index.

Even today, portfolio managers who are employed in mutual funds are usually measured against some cap-weighted benchmark. These benchmarks are usually indexes that cover the majority of a market capitalization in that country or continent. In the United States, portfolio managers performances can be measured against S&P 500 index while in Europe portfolio managers can be measured against index such as the STOXX Europe 600.

Based on that, it is clear that cap-weighted index presents a good starting point for comparison of different weighting approaches. Researchers who proposed new weighting schemes all compared the performance against cap-weighted index. This is the right thing from theoretical standpoint and it is something that is also done by mutual funds. Thus, I will calculate the weight of an individual stock in the cap-weighted portfolio with the following formula:

$$w_i = \frac{Market\ capitalization\ of\ stock\ i}{Total\ market\ capitalization\ of\ portfolio\ stocks} \tag{10}$$

If the CAPM does not hold and it is possible to find overpriced or underpriced stocks, then the market-cap weighting will be an inefficient strategy and investors will construct portfolios that overweight the overpriced stocks and underweight the underpriced stocks (Arnott, Kalesnik, Moghtader & Scholl, 2010). That was identified as a main drawback of the market-cap weighted indexes by many portfolio managers because a portfolio can exhibit huge gains in the bull markets when market excesses build up, and subsequently be followed by huge losses when stock prices revert to their fair values. The fact that cap-weighted indexes perform the best in strong bull markets can also be recognized as a growth bias (Arnott, Hsu & Moore, 2005).

## 2.2 Equally weighted portfolio

The first alternative to the market-cap weighted portfolio is equally weighted portfolio (hereinafter: EW portfolio), where I assign equal weight to each company in a portfolio, regardless of a size of a company. This approach does not require any estimates of expected returns, variances or covariance matrix nor does it give any information value to all private or public information about a company. This approach basically says that investors have zero ability to forecast anything. The equally weighted stock market indexes are composed of the same companies as the cap-weighted indexes except that I assign the same weight to each company, e.g. S&P 500 Equal Weight Index (Arnott, Kalesnik, Moghtader & Scholl, 2010). The formula for equal weighting can be expressed as follows:

$$w_i = \frac{1}{Number\ of\ stocks\ in\ a\ portfolio\ (N)} \tag{11}$$

Contrary to the market-cap weighting where one can overweight overvalued and underweight undervalued stocks, the equal weighting approach holds small and large companies in the same proportions. This strategy can result in higher transaction costs and lower capacity for investment in such indexes which can be problematic for large institutional investors who manage huge sums of money (Arnott, Kalesnik, Moghtader & Scholl, 2010).

But there is also strong empirical evidence that equally weighted portfolios outperform cap-weighted ones. Choueifaty and Coignard (2008), Clarke, De Silva and Thorley (2013) confirmed that when they tested multiple weighting strategies while Plyakha, Uppal and Vilkov (2012; 2014; 2015), Malladi and Fabozzi (2017) found that in a direct comparison of both approaches. Similarly, DeMiguel, Garlappi and Uppal (2009) compared equally weighted portfolio strategy with the mean-variance model and its 14 extensions and came to the conclusion that none of the models is consistently better than naive diversification (equal weighting) in terms of Sharpe ratio, turnover or certainty-equivalent return but the research was based on the sector and country portfolios.

Plyakha, Uppal and Vilkov (2012) compared performance of the equal-weighted portfolio with cap-weighted and price-weighted portfolio (a price-weighted portfolio is weighted by price per share, example of such index is Dow Jones Industrial Average). They found that equal-weighted portfolio outperforms both approaches in terms of average return, Sharpe ratio, four-factor alpha and certainty-equivalent return. On the other hand, equally weighted portfolio had higher volatility and turnover but even after taking the transaction costs into account, it outperformed the other two approaches. Plyakha, Uppal and Vilkov (2015) found that difference in total mean returns accounted to 2.71% points per year for equal weighted portfolio when compared with cap-weighted. With the four factor model, they decomposed total return in systematic component and alpha. They found that systematic return of equal weighted portfolio is higher because of higher exposure to market, size and value factors – these are factors that have been identified as sources of higher returns by Fama and French (1993) but it had negative exposure to momentum factor – factor identified by Carhart (1997). With regard to higher alpha, they found that it arises from the monthly rebalancing which is needed to maintain equal weights. When they reduced rebalancing to six and twelve months, alpha was statistically indistinguishable from that of the cap-weighted portfolio. The findings of Plyakha, Uppal and Vilkov (2012; 2014; 2015) were confirmed by Malladi and Fabozzi (2017) based on the real world data and simulations. They concluded that equal weighting makes economic sense even after they accounted for higher portfolio turnover costs.

### 2.3 Fundamentally weighted portfolio

Arnott, Hsu and Moore (2005) proposed to weight stocks by their fundamentals, such as book values, cash flows, revenues, sales, dividends and total employment. They argue that cap-weighted indexes are sub-optimal when there is price inefficiency in the market and this should be corrected by weighting portfolio stocks by their fundamentals (Hsu, 2006; Treynor, 2005). They say that fundamentally weighted portfolio (hereinafter: FW portfolio) should preserve the benefits of cap weighting, i.e. they should be mostly concentrated in large-cap stocks thus preserving investment capacity and liquidity and should have low transaction costs.

The measures used for construction of fundamental indexes by Arnott, Hsu and Moore (2005) were the following:

1. Book value,
2. Trailing five-year average revenue,
3. Trailing five-year average gross sales,
4. Trailing five-year average cash flow,
5. Trailing five-year average gross dividends, and
6. Total employment.

The companies were first ranked by each metric and then the 1,000 largest were included in the index at its relative metric weight. Additionally, they constructed composite index where they excluded total employment because the numbers are not always available, and revenues, which are highly correlated with sales. Composite index weights were determined by equally combining weights that each company would have in four fundamental indexes. Then they selected the 1,000 largest companies and weighted them by composite weight. All portfolios were rebalanced annually after the last trading day of each year based on the most recent financial data. When they tried to rebalance index more frequently, there were no benefits in terms of higher returns but just higher index turnover (Arnott, Hsu & Moore, 2005).

The results obtained for the fundamental portfolios in the US stock market from 1962 through 2004 were compared with a cap-weighted portfolio composed of 1,000 stocks. On average, fundamental indexes had 1.97% points higher return than the S&P 500 portfolio, 2.15% points higher return than the cap-weighted benchmark and they also outperformed equal weighted S&P 500 with lower risk. The Sharpe ratio was also higher for fundamental indexes. After they accounted for transaction costs, excess return of fundamental indexes fell to 2.01% points. Out of all the fundamental indexes, the sales index performed the best as it realized excess return of 2.42% points after transaction costs. The period when fundamental indexes performed worse than the cap-weighted index was only during strong bull markets, especially in years 1998-1999, when large-cap companies experienced huge growth. But fundamental indexes more than compensated for lagged performance in the years that followed as their outperformance was especially strong in bear markets while in average bull market fundamental indexes kept pace with cap-weighted ones. When it comes to sector weightings, the fundamental indexes change composition more slowly as the economy evolves, while cap-weighting can quickly increase exposure to sectors favoured by investors (Arnott, Hsu & Moore, 2005).

Hsu and Campollo (2006) compared performance of fundamental indexes and MSCI cap-weighted indexes for 23 countries from 1984 to 2004 and found average outperformance of fundamental indexes by 2.8% points. The outperformance was robust over different market conditions and also held for small and medium sized companies. These indexes also had slightly lower standard deviation and average beta lower than one. Again, fundamental indexes underperformed only at the height of the dot-com bubble. Hemminki and Puttonen (2008) tested fundamentally weighted portfolio using European data between 1996 and 2006. They constructed fundamentally weighted portfolios from 50 large-cap European stocks. The fundamental portfolios average excess return was 1.76% points per year with all portfolios producing higher returns and higher Sharpe ratios. However, results were statistically significant only for dividend portfolio, book value portfolio and composite portfolio which can be explained by relatively short observation period.

Why fundamentally weighted portfolios outperform cap-weighted ones has led to a debate between researchers. Hsu and Campollo (2006), Arnott, Hsu and Moore (2005) say that fundamental weighting portfolio performs better because it increases weight of the company only when it grows book value, cash flows, sales and dividends faster than the rest of the economy. Thus, it underweights growth companies that are not increasing their fundamentals. They also disagree that fundamental weighting is actually value strategy and argue that fundamentals strategy outperforms value strategy in all market conditions while it preserves investment capacity and diversification. Fundamental weighting also retains some exposure to growth companies that are increasing their fundamentals in contrast to equal weighting which discards them.

On the other hand, Perold (2007) and Kaplan (2008) disagree with the proponents of fundamental weighting and say that stock market prices fluctuate around their fair values, and call their line of reasoning the "noisy market hypothesis". Perold (2007) showed that the pricing error is uncorrelated with the stock's fair value and is also uncorrelated with the stock's market value. This shows that cap-weighting does not necessarily underweight undervalued stocks and overweight overvalued stocks.

Thereby, the superior performance of fundamental weighting is a result of active portfolio management and investment in value stocks. Value-biased portfolios have historically outperformed unbiased portfolios and it should not be a surprise that fundamentally weighted portfolios also outperform cap-weighted ones. That is the same reason as why equally weighted portfolios do better than cap-weighted ones. And despite their elegance, fundamental indexes may not be the most efficient way to capture value premium as there are possibly more advanced multi-factor strategies that will exploit other anomalies (Perold, 2007; Kaplan, 2008; Blitz & Swinkels, 2008).

### 2.4 Minimum variance portfolio

The approaches that I described so far are simple to implement and do not require any estimates of expected returns, variances and covariance matrix. The market-cap weighting comes from financial theory, while equal weighting and fundamentals weighting are considered heuristic approaches. With minimum variance portfolio (hereinafter: MV portfolio), I return back to the theoretical grounds of the modern portfolio theory.

Minimum variance or global minimum variance portfolio is the most left point on the efficient frontier as it was shown in Figure 3. To construct minimum variance portfolio, one has to optimize portfolio weights using only a variance-covariance matrix. Minimum variance portfolios are found by equalizing the marginal risk contributions of each stock to portfolio risk (Clarke, De Silva & Thorley, 2013). This is different than finding a portfolio with the highest risk-reward ratio (the maximum Sharpe ratio portfolio) where one must optimize portfolio using variance-covariance matrix and expected returns.

The minimum variance portfolio is thus considered risk-based approach as it optimizes portfolio only with respect to risk measure. Risk-based approaches became popular because it is easier to estimate variances and covariance matrix than expected returns. If expected returns are estimated with high imprecision, this can result in very poor portfolio optimization (the problems of estimation of portfolio moments are covered in detail in the third chapter). As a consequence, researchers studied risk-based approaches extensively and proposed other approaches to portfolio optimization such as an equally-weighted risk contributions portfolio and most-diversified portfolio which I will present after the minimum variance portfolio.

The global minimum variance portfolio and different methods for forecasting covariance matrix were studied extensively by Chan, Karceski and Lakonishok (1999). In their study for the US equity market between 1973 and 1997, the minimum variance portfolio annualized standard deviation was 12.94% compared to 16.62% for the equally weighted portfolio and 15.54% for the cap-weighted portfolio. Sharpe ratios were 0.64, 0.60 and 0.45, respectively, and the average betas of different optimized minimum variance portfolios were between 0.5 and 0.7 compared to 1.07 for the equally weighted portfolio. The minimum variance portfolio invested more than 40% to utility sector while the cap-weighted portfolio invested 8.66% and the equally-weighted portfolio invested 15.31% on average. The minimum variance portfolios were also slightly tilted towards the larger stocks and value stocks. They concluded that minimum variance portfolio helps reduce risk, even though the correlation between covariance forecasts and realized covariances was only 0.20.

Clarke, De Silva and Thorley (2006) examined minimum variance portfolio performance in the US from 1968 till 2005. They took constraints that most portfolio managers face into account and did not allow short sales and limited maximum weight for individual stock. They found similar pattern as Chan, Karceski and Lakonishok (1999) as cap-weighted benchmark had annualized return of 11.5%, standard deviation of 15.4% and Sharpe ratio of 0.36, compared to the minimum variance portfolio which had 12.4%, 11.7% and 0.55, respectively. In both studies, portfolio variance was reduced between 20% and 25%, but what is more interesting is that minimum variance portfolios still realized higher returns which resulted in much higher Sharpe ratios. Clarke, De Silva and Thorley (2006) explain the higher return of minimum variance portfolios with higher exposure to low-cap and value stocks.

The logic of minimum variance portfolios outperforming cap-weighted portfolios is not consistent with the CAPM and the established financial theory. How can minimum variance portfolios offer lower volatility but at the same time exhibit returns that match or exceed the market? This anomaly of high risk stocks not necessarily offering higher returns has been recorded by Fama and French (1992), Ang, Hodrick, Xing and Zhang (2006) further described this anomaly of low risk stocks having high returns and Clarke, De Silva and Thorley (2011) showed how minimum-variance portfolios exploit this long-standing critique of the CAPM and load portfolios with low beta stocks which lower portfolio risk but maintain a decent return.

The research from Chow, Hsu, Kuo and Li (2014) tested minimum variance portfolios in the global and emerging markets. The minimum variance portfolios in the global and emerging markets had 30% and 50% lower volatility than the cap-weighted portfolios. This is not surprising as correlations in the US market are higher and potential benefits of minimum variance portfolios are higher for the global and emerging markets. Improvement in Sharpe ratio was also statistically significant. Therefore, the anomaly is consistent across countries and time. Additionally, they compared optimized minimum variance portfolios with simple heuristic approaches such as weighing portfolio by inverse of beta and inverse of volatility. They found optimized portfolios to have lower volatility but also lower long-term returns which resulted in similar Sharpe ratios for all strategies.

Minimum variance portfolios should thus be considered by investors as possible alternatives that can yield decent return at low risk. However, as Chow, Hsu, Kuo and Li (2014), Clarke, De Silva and Thorley (2006) noted there are some drawbacks such as high tracking errors relative to the cap-weighted portfolios which can be problematic for portfolio managers who are evaluated against cap-weighted benchmarks. The cap-weighed portfolio can strongly outperform minimum variance portfolio in bull markets. Additionally, minimum variance portfolios also have slightly lower investment capacity, lower liquidity and higher turnover but this should not discourage portfolio managers to consider investing in minimum variance portfolios.

## 2.5 Equally weighted risk contributions portfolio

Related approach to minimum variance portfolio is risk parity or equal risk contribution portfolio (hereinafter: ERC). The equally weighted risk contributions portfolio was first mentioned by Qian (2005), who proposed to allocate risk equally across asset classes, which is very different than balanced allocation of capital. He described this approach as risk parity and showed on historical data how in a 60/40 portfolio split between stocks and bonds, stocks contributed 93% of risk and bonds contributed 7%. This clearly shows that capital allocation is not equal to risk allocation.

Maillard, Roncalli and Teiletche (2010) researched equally weighted risk contributions approach applied to sectors portfolio. They describe the ERC approach as the middle ground between equally weighted and minimum variance portfolio. On one hand minimum variance portfolio can suffer from high concentration (I described that in the previous chapter where I mentioned huge concentration in low volatility sectors such as utilities) and on the other hand equally weighted portfolio can have poor risk diversification if there are significant differences in individual risks. Equally weighted risk contributions portfolio should solve these problems as it equalizes risk contributions from different portfolio components, which maximises diversification of risk. This means each asset or stock contributes the same amount towards the total portfolio risk. Qian (2006) showed that risk contributions can be good predictors of large losses. That is why the ERC strategy can prevent enormous losses in a 2007/2008 market environment.

In the ERC portfolio, each stock’s total risk contribution is equalized compared to the minimum variance portfolio which equalizes risk contribution on the marginal basis. Thus, the ERC portfolios will lie within efficient frontier (Clarke, De Silva & Thorley, 2013) and stocks with higher volatility will have lower weights and vice versa. The ERC portfolio can also be constructed using different risk measures such as value-at-risk or expected shortfall (Stefanovits, 2010).

Maillard, Roncalli and Teiletche (2010) showed theoretically and empirically that ERC portfolios have volatility between the minimum variance and the equally weighted portfolios. Theoretical exercise clearly showed how the ERC portfolio weights are ranked in the same order as in the minimum variance portfolio but they are more balanced. The ERC portfolio thus represents a variance minimizing portfolio with a constraint of sufficient diversification. Additionally, when they compared the ERC portfolio with the minimum variance portfolio and the equally weighted portfolio, they found that ERC portfolio performed the best in terms of Sharpe ratio when there was huge heterogeneity in correlation coefficients and individual volatilities, which was the case for the global diversified portfolio. On the other hand, Sharpe ratio was in line with equally weighted strategy for the US sectors portfolio as there was similarity in correlation coefficients and volatilities, the minimum variance strategy had the highest Sharpe ratio in that sample. All in all, the ERC portfolio can be considered as an alternative portfolio construction method as it has smaller drawdowns, is less concentrated and has lower turnover than compared strategies.

Others, who tested the efficiency of the ERC portfolio, have mostly confirmed the findings. Stefanovits (2010) reported that portfolio constructed from 500 stocks where risk was similarly distributed did not outperform 1/N strategy. Cagna and Casuccio (2014) reported favourable results when using expected shortfall as risk measure and its benefits for protection against large losses. The ERC approach can yield the best results when it is applied to sector portfolios that are already diversified by market-cap, equal weighting or some other approach - the fields were the approach was also researched the most - as an asset class allocation strategy.

### 2.6 Most diversified portfolio

The last of the risk based approaches presented is the most diversified portfolio (hereinafter: MD portfolio), which was first introduced by Choueifaty and Coignard (2008). They defined the most diversified portfolio as a portfolio that maximizes the diversification ratio. This is the ratio of the weighted average of volatilities divided by the portfolio volatility, which can be written as:

$$D_P = \frac{w^T * \sigma}{\sqrt{w^T * \sum * w}} \tag{12}$$

Where σ is N*1 vector of volatilities, ∑ is a covariance matrix and $D_P$ is the diversification ratio which will be strictly higher than 1 for long-only portfolio, except when portfolio consist of only one asset in which case the diversification ratio will be equal to 1.

The diversification ratio of the most diversified portfolio will increase when the correlations decrease. In the extreme case when all securities are perfectly correlated, the diversification ratio will be equal to 1. In the most diversified portfolio, all stocks included in the portfolio have the same correlation to the portfolio, while any other stock not included in the most diversified portfolio has higher correlation (Choueifaty, Froidure & Reynier, 2013).

Choueifaty, Froidure and Reynier (2013) favour the most diversified portfolio over the other risk-based approaches because the minimum variance portfolio invests only in asset with low volatility which makes the ERC portfolio and the most diversified portfolio the only portfolios that are balanced on the risk contribution basis. But the ERC portfolio invests in all stocks which is not always the best thing. To test the approach, Choueifaty and Coignard (2008) compared the most diversified portfolio with a market-cap weighted portfolio, equally weighted portfolio and minimum variance portfolio. They analysed performance in the US and the Eurozone equity markets from 1992 to 2008 and found superior risk adjusted returns for the most diversified portfolio in both regions. The most diversified portfolio had the highest return of all portfolios and variance that was smaller only for the minimum variance portfolio. The cap-weighted benchmark performed the worst in terms of Sharpe ratio and showed very poor performance in that period.

Choueifaty, Froidure and Reynier (2013) added the ERC portfolio to comparison. They found all portfolios to outperform cap-weighted portfolio with minimum variance portfolio, ERC portfolio and most diversified portfolio having higher returns and lower volatility than cap-weighted benchmark. Equally weighted portfolio had higher returns than cap-weighted portfolio but similar volatility. Methods that used asset covariance matrix resulted in better diversification and lower volatility. The comparison of risk based approaches showed minimum variance and most diversified portfolio to be the best candidates for optimal portfolio as they successfully minimized volatility and had high Sharpe ratio.

Testing the exposure to Fama and French factors Choueifaty, Froidure and Reynier (2013) found all approaches to have positive exposure to the SMB factor, which was the highest for equally weighted portfolio. Risk based approaches had the lowest exposure to market factor and all approaches were also exposed to the HML factor which measures bias towards value stocks. Finally, alpha was the highest for the most diversified portfolio which is consistent with the goal of maximum diversification having balanced exposure to risk factors.

In contrast, the study from Clarke, De Silva and Thorley (2013) tested the same approaches - market-cap weighting, equal weighting, minimum variance, equal risk contributions and maximum diversification. The research was done for the largest 1,000 US stocks from 1968 to 2012. Their results were very different than those of Choueifaty and Coignard (2008), Choueifaty, Froidure and Reynier (2013) as they reported the highest Sharpe ratio for minimum variance and equal risk contributions portfolio. High Sharpe ratio was also reported for equal weighting while maximum diversification had Sharpe ratio lower than market-cap weighted portfolio. Excess annual return was the highest for equally weighted and risk parity portfolio and volatility was the lowest for the minimum variance portfolio. Surprisingly, the most diversified portfolio had the highest volatility.

Another important thing is the average number of positions. Both maximum diversification portfolio and minimum variance portfolio included on average less than 100 securities. This implies that by selecting fewer less correlated and less risky securities it is possible to achieve better risk reduction (Clarke, De Silva & Thorley, 2013). Additionally, if anomaly of low risk and high return documented by Ang, Hodrick, Xing and Zhang (2006) will persist in the future then low volatility construction methods such as the minimum variance could also offer higher returns.

To conclude, risk based approaches offer great opportunities for reducing portfolio risk, but approaches such as equal risk contributions and maximum diversification are relatively new and are not well documented. Many researchers point out that results for different weighting schemes may be subject to data mining and are difficult to replicate. This can also be the case for the most diversified portfolio which performed very differently in the studies of Choueifaty and Coignard (2008), Choueifaty, Froidure and Reynier (2013) and Clarke, De Silva and Thorley (2013). The equal risk contributions portfolio or risk parity is also relatively new approach first introduced in the 1990s as an asset allocation strategy and was applied to stock portfolios only recently. Therefore, results from minimum variance portfolio may be more robust as it was already detailed in the modern portfolio theory in the 1960s (Clarke, De Silva & Thorley, 2013).

## 2.7 Maximum Sharpe ratio portfolio

Maximum Sharpe ratio portfolio (hereinafter: MSR portfolio) is the tangency portfolio from the modern portfolio theory as it was defined by Markowitz (1952). It is the portfolio with the highest return per unit of risk, at least on an ex ante basis. In theory, the construction of optimal risky portfolio should be a goal of every portfolio manager. Investor’s risk aversion should come into play only in the second step when funds are split between risky portfolio and risk-free asset – the process described by Tobin (1958).

Markowitz (1956) derived the mean-variance model and showed exactly how investors should maximize expected return for every level of risk. But the process of finding optimal risky portfolio includes estimation of stock expected returns and variance-covariance matrix. I already mentioned difficulties in estimating covariance matrix, which is the only input for risk based portfolio construction methods, but expected returns are even more difficult to estimate. Merton (1980) showed how small changes in expected returns can lead to very different portfolio weights. This was later observed also by other researchers and in the absence of good input estimates Michaud (1989) described the process as error maximization.

DeMiguel, Garlappi and Uppal (2009) tested 14 models for mean-variance optimization and concluded that none of them beats naïve equally weighted strategy on a consistent basis due to the errors in estimating portfolio parameters. This lead Martellini (2008) to design optimal risky portfolios based on the improved estimators of expected returns and variance-covariance matrix. The research was extended two years later by Amenc, Goltz, Martellini and Retkowsky (2010). To construct the MSR portfolio they used the stock's total downside risk as a proxy for the expected excess return and principal component analysis to estimate the covariance matrix (I describe the methods in detail in the next chapter).

The results obtained by Martellini (2008), when using improved estimators for expected returns and variance-covariance matrix, showed it is possible to achieve superior risk-adjusted returns with the MSR portfolio. He found that market-cap weighted portfolios had the lowest Sharpe ratios and were also outperformed by equally weighted counterparts and the MV portfolios. The comparison of equally weighted and minimum variance portfolios suggested that focusing only on volatility results in lower risk but severely affects performance of the MV portfolios. Turning to the MSR portfolios, he found they outperformed all other portfolios when he used total volatility as a measure of expected returns in combination with factor based estimator of the correlation structure of stock returns. When he used other correlation estimators or different models for estimating expected returns, such as the Fama-French model, he found that the MSR portfolios deliver similar Sharpe ratios as equally weighted counterparts. Thus, it is possible to construct better equity benchmarks with maximum Sharpe ratio portfolios, but their efficiency depends heavily on the estimators used for the expected returns and correlation structure.

Amenc, Goltz, Martellini and Retkowsky (2010) extended the preliminary research done by Martellini (2008). They adjusted the method of estimating expected returns to control higher order portfolio moments and used stock's semi-deviation instead of total volatility. The covariance matrix estimation process was also adapted by using the principal component analysis instead of forcing predetermined factor structure. The efficient MSR portfolio was compared with the market-cap weighted S&P 500 index from 1959 to 2008. The MSR portfolio was constructed to have the same constituents as the benchmark index by imposing minimum and maximum weight constraints. Short sales were prohibited and weights had to sum to one. They also implemented approach to control for portfolio turnover and did not necessarily rebalance the portfolio every quarter unless there was significant deviation from optimal weights. These constraints resulted in a more balanced portfolio and lower transaction costs which would need to be as high as 13% to offset the benefits of efficient diversification.

The obtained results showed efficient weighting to have lower volatility, higher average return and higher Sharpe ratio. The Sharpe ratio was roughly 70% higher and the difference was statistically significant at 0.1% level. The difference in annualised return was also confirmed by the CAPM analysis which showed higher alpha for efficient diversification and confirmed that the MSR portfolio did not have greater exposure to market risk. The higher Sharpe ratio also did not result in higher value-at-risk or downside risk. The comparison with equally weighted portfolio again showed superiority of the MSR portfolio although less pronounced than with respect to the cap-weighted portfolio. The

Sharpe ratios were 0.41 for the MSR portfolio, 0.35 for the EW portfolio and 0.24 for the cap-weighted portfolio (Amenc, Goltz, Martellini & Retkowsky, 2010).

An overview of efficient indexation in different economic conditions, time periods and market conditions showed greater stability of returns than the cap-weighted portfolio and much higher ending value of one dollar invested over the 49 years period. The only period when the cap-weighted portfolio delivered superior returns was the strong bull markets of the late 1990s when it was difficult to outperform the trend following strategy. Otherwise, the MSR portfolio was more efficient as Sharpe ratios were consistently higher than those of the cap-weighted portfolio in recessions, expansions and in the periods of low and also increased market volatility. International evidence for other regions (United Kingdom, Eurobloc, Japan and Asia-Pacific ex Japan) in the approximately seven years period between 2002 and 2009 showed improved Sharpe ratios for all countries/regions. This suggests that using efficient diversification should also yield similar results in other stock markets (Amenc, Goltz, Martellini & Retkowsky, 2010).

# 3 ESTIMATION OF PORTFOLIO MOMENTS

In this chapter, the focus is on portfolio moments and review of different methods for their estimation. Expected returns and variance-covariance matrix present first and second portfolio moment, respectively. Higher order moments include coskewness and cokurtosis matrix which present third and fourth portfolio moment, respectively.

In the context of the modern portfolio theory, I am looking for a mean-variance optimal portfolio and assume normal distribution of stock returns. Thus, the portfolio is optimized only with respect to the first and second portfolio moment. However, when stock returns deviate hugely from the normal distribution the optimization techniques that optimize portfolio with respect to the third and fourth portfolio moment can improve portfolio risk adjusted return.

## 3.1 Expected returns

In literature, it is well documented that sample based means are bad proxies for stock's future expected returns as they result in poor portfolio optimization. This has been shown by Merton (1980), Chopra and Ziemba (1993), Britten-Jones (1999) and many others, who showed that small differences in expected returns will affect optimizers to allocate the largest weights to stocks with the most extreme estimates of expected returns – large long positions in stocks with high expected returns and large short positions in stocks with negative expected returns. In the absence of an efficient estimation of expected returns, heuristic and risk based approaches were developed which do not rely on expected return estimates.

But to construct the tangency portfolio as defined in the modern portfolio, theory efficient proxies for expected returns are needed. There are several methods that can be used to estimate expected returns and improve the out-of-sample performance of the MSR portfolio. I will briefly present the next approaches:

1. Factor models,
2. Shrinkage estimators,
3. Black-Litterman model,
4. Resampling,
5. Implied cost of capital, and
6. Volatility measures.

The first group of methods for estimating expected returns include factor models, i.e. the CAPM, Fama and French 3-factor model, its extension with momentum factor and Fama and French 5-factor model. I described these models in the first chapter of the thesis where I said models with more factors have the potential to yield better results than the CAPM. However, empirical evidence of portfolio diversification showed that factor models may not be the best for optimizing portfolios. The study from Bartholdy and Peare (2005) compared the CAPM and Fama and French 3-factor model for estimating expected returns and concluded that 3-factor model is marginally better but it can still explain only 5% of differences in returns. Martellini (2008) reported lower Sharpe ratios for optimal portfolios when using 3-factor model instead of the total volatility measure. Additionally, Fama and French (1997) themselves showed that large standard errors are typical for the CAPM and for the 3-factor model due to the uncertainty about true factor risk premiums.

Shrinkage concept was first introduced by Stein (1955), who demonstrated the benefit of shrinking sample estimates toward some constant. Today, one of the most popular shrinkage estimators for expected returns is the Bayes-Stein estimator. The idea is that each asset's expected return should be shrunk toward a common value or some global mean. This common value can be set to equal the expected return of minimum variance portfolio (Jorion, 1986). As it has been showed by Jorion (1985; 1986), this reduces estimation risk and improves portfolio performance. The improvement in performance is achieved by reducing extreme expected return estimates and thus big portfolio weights, which suggests that most of the diversification benefits come from risk reduction.

The Black-Litterman model was developed by Goldman Sachs employees (Black & Litterman, 1992), who derived a model that starts with the equilibrium return of stocks defined by the global CAPM. The weights implied by equilibrium returns can then be tiled according to investor's beliefs about stocks relative or absolute performance. Additionally, investor can specify how strong are his views about stock performances and thus the influence on portfolio weights. This approach starts with balanced (market-cap) weights and then increases weights of those stocks favoured by portfolio manager.

Another way to improve portfolio optimization was proposed by Michaud and Michaud (1998) called resampling and is based on Monte Carlo methods. The resampling approach first resamples the data and let investors create hundreds of alternative efficient frontiers, which are then averaged to arrive at a resampled efficient frontier with average weights (Michaud & Michaud, 2008). The approach received mixed support as Markowitz and Usmen (2003) found that on average Michaud resampled frontier achieves higher utility than Bayesian approach while Harvey, Liechty and Liechty (2008) reported the contrary and rejected the findings of Markowitz and Usmen (2003).

Additional approach which can be used to improve upon expected return estimates is the use of implied cost of capital. The adequacy of this approach was tested by Pastor, Sinha and Swaminathan (2008), who confirmed implied cost of capital as a useful tool for capturing time-varying expected returns and its positive correlation with risk. Bielstein and Hanauer (2018) used implied cost of capital based on analyst's earnings forecasts as a proxy for expected returns in portfolio construction. Implied cost of capital equates forecasted cash flows to equity with the current stock price. Additionally, they corrected predictable errors in analysts forecast and used adapted expected return estimates in mean-variance optimization framework. They reported favourable results using these estimates as their MSR portfolio outperformed cap-weighted, equally weighted and minimum variance portfolio. The drawback of this approach is the need for sufficient analyst's coverage of stocks, which can limit the usefulness of the approach to large cap stocks and developed markets.

The last approach includes use of volatility measures as a proxy for excess expected returns. Total volatility as a proxy for excess expected returns and input to portfolio optimization was first used by Martellini (2008), who followed the evidence that idiosyncratic volatility has explanatory power for the cross section of expected returns. This was shown by Malkiel and Xu (2006), who demonstrated that if investors are unable to hold market portfolio for whatever the reason, then they will also care for idiosyncratic risk (risk of particular asset). This is contrary to the popular belief and traditional models such as the CAPM which says that only market risk should be priced. This means stocks with higher firm-specific risk should earn higher returns, something that was also confirmed by Malkiel and Xu (1997).

The estimate of total volatility, which is the sum of market and firm-specific risk, resulted in superior performance of the MSR portfolio in a study by Martellini (2008). He also showed that explanatory power of total volatility for predicting excess expected returns is not driven by systematic volatility. The total volatility estimate was slightly modified in the later research by Amenc, Goltz, Martellini and Retkowsky (2010) to take into account also higher order moments. They used stock's semi-deviation which considers only deviations below the mean. The performance of the MSR portfolio using semi-deviation as a measure of excess expected returns resulted in much better performance than a cap-weighted portfolio.

### 3.2 Variance-covariance matrix

Despite the consensus among researchers that errors in expected returns have bigger impact for portfolio optimization than errors in return variances and covariances, the later are also important and have to be estimated precisely (Merton, 1980; Chopra and Ziemba, 1993; Chan, Karceski & Lakonishok, 1999).

When estimating vector of returns, the number of estimates needed is the same as the number of stocks in a portfolio. The same number of estimates is needed for return variances, while number of return covariances is much higher. The covariance matrix (or correlation matrix) for a portfolio consisting of 50 stocks will need 1,225 correlation estimates (=50 x 49/2). The example of the correlation matrix for portfolio of 4 stocks is presented below:

$$\begin{bmatrix} 1 & - & - & - \\ 0.2 & 1 & - & - \\ 0.7 & 0.9 & 1 & - \\ 0.5 & -0.1 & 0.8 & 1 \end{bmatrix} \tag{13}$$

As one can see in the above correlation matrix some correlations are very low (even negative) and this can pose a problem for portfolio optimization. Maybe the correlations between stocks are in reality that low and one would be able to construct portfolio with very low volatility but usually this is not the case as too low correlations are mostly the result of an estimation error. Chan, Karceski and Lakonishok (1999) attributed poor performance of sample covariance matrix to a possibility of firm-specific events that affected several stocks, but these events are not expected to continue in the future. That is why sample correlation matrix should not be used in portfolio optimization as one can get very extreme weights.

The correlation matrix estimation process can be improved with several methods. I will present the following methods which are most widely documented and tested:

1. Portfolio constraints,
2. Factor models,
3. Principal component analysis,
4. Shrinkage estimators, and
5. Use of high frequency (daily) data.

One simple way of reducing estimation error in covariance matrix is imposing no-short-sale constraints. This was found by Jagannathan and Ma (2003) who showed that when one do not allow negative weights, sample covariance matrix performs in line with a covariance matrix estimated based on shrinkage estimators, factor models and daily data. The reason why non negativity constraint helps is the fact that stocks which have high correlation with others will usually receive negative weights (short positions). But high correlation can be subject to upward biased estimation error which is reduced by forcing non negativity constraint. The similar logic applies to using upper bound constraints. Stocks which have low correlation will in normal circumstances receive huge weights as they reduce portfolio risk, but this can be result of downward biased estimation error. However, when non-negativity constraints are already in place additional restrictions such as upper bound weight limits do not offer additional significant improvement. On the other hand, constraining portfolio weights does not affect errors in expected returns.

The second group are factor models which were also widely used in the research papers. Chan, Karceski and Lakonishok (1999) found some underlying structure in return covariances and showed that factor models better capture this structure. They tested the standard CAPM model, the 3-factor Fama and French model, its extension with momentum factor and also an eight-factor model and a ten-factor model. They found factor models to be successful in levelling out the covariances, which resulted in less extreme forecasts. However, a few factors such as market, size and book-to-market value factors are enough to help capture a covariance structure, and adding more factors do not necessarily reduce a forecast error as these models tend to overfit the data. Favourable out of sample result using factor-based correlation matrix with Fama and French 3-factor model was obtained by Martellini (2008) and Chow, Hsu, Kuo and Li (2014).

A large number of researchers also use principal component analysis (hereinafter: PCA) which extracts common factors that are driving the co-movement of stock returns. The principal component analysis is in fact a factor model, but with a difference that it determines the underlying risk factors from the data. This is different than the classical factor models where one have to choose the model with the right number of factors. The main benefit of PCA is thus to extract the factors which explain most of the variability of stock returns without relying on the particular factor model to be a true pricing model (Amenc, Goltz, Martellini & Retkowsky, 2010). Success of the PCA was reported in studies from Fujiwara, Souma, Murasato and Yoon (2006), Clarke, De Silva and Thorley (2006), Amenc, Goltz, Martellini and Retkowsky (2010) and Chow, Hsu, Kuo and Li (2014) to name a few.

The next method is shrinkage of the sample covariance matrix. I will describe the approach derived by Ledoit and Wolf (2003; 2004) which is similar to the Bayes-Stein shrinkage estimator used for estimating expected returns. The shrinkage estimator for covariance matrix uses sample covariance matrix and pulls the extremely high coefficients downwards to compensate for the likely positive estimation error, and vice versa. It pulls extremely low estimated coefficients upwards to compensate for negative estimation error. The shrinkage target towards which sample covariance matrix is shrunk is a single factor model (the CAPM) in Ledoit and Wolf (2003) and constant correlation model in Ledoit and Wolf (2004). The two shrinkage targets have comparable performance but the latter is easier to implement. In the tests performed by Ledoit and Wolf (2004), sample covariance matrix performed the worst while principal components matrix produced similarly good results as shrinkage estimators.

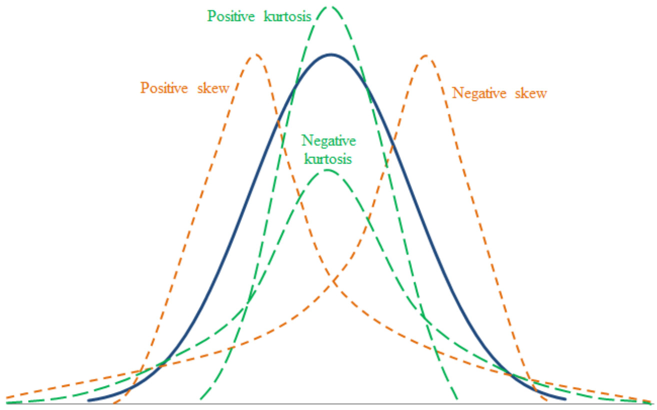
Using daily returns to estimate covariance matrix instead of weekly or monthly is also a widely used approach by practitioners and academics. Daily data can be used to estimate daily sample covariance matrix or daily covariance matrix based on factor models, such as one-factor model and three-factor model (Jagannathan and Ma, 2003). When using daily data, one increases the number of observations which has a potential to improve the prediction of covariances (Gosier, Madhavan, Serbin & Yang, 2005).

When it comes to forecasting variances Chan, Karceski and Lakonishok (1999) found that variances are more stable and thus easier to predict. One can, therefore, use sample estimates of past variances as they give relatively good prediction of future variances. One way to improve variance prediction is factor models, but again models with more factors do not necessarily raise forecasting power (Chan, Karceski & Lakonishok, 1999). Sample estimates were used in a study by Martellini (2008) and performed quite well. Additionally, Maretllini (2008) proposed to use a simple GARCH (1, 1) model to account for the presence of volatility risk as there is extensive empirical evidence that stock market volatility changes randomly over time. The result was somewhat improved performance of the tangency portfolio but the difference was not huge.

### 3.3 Coskewness and cokurtosis matrix

Markowitz mean-variance framework requires only expected returns and variance-covariance matrix as the inputs to portfolio optimization. But stock returns are not always normally distributed, which is the assumption of the modern portfolio theory. Optimizations are usually done without controlling for skewness and kurtosis but introduction of the estimators which control also for the skewness and kurtosis can improve the out of sample performance of optimized portfolios. The Figure 7 shows how skewness and kurtosis affect the normal distribution.

Figure 7: Skewness and kurtosis




*Source: Own work.*

In the above figure, one can see standard normal distribution, negatively and positively skewed distribution, leptokurtic distribution (positive kurtosis) and platykurtic distribution (negative kurtosis). Skewness is a measure of asymmetry of the distribution with the value of zero for normal distribution. If skewness coefficient is lower than zero, then there is negatively skewed distribution, and vice versa. Negative skewness is undesirable for investors as it has longer and fatter tail on the left side, which means higher probability for larger losses. On the other hand, kurtosis is a measure of tailedness of the distribution, where the value of the kurtosis coefficient for normal distribution equals 3. For platykurtic distribution (positive kurtosis) value of kurtosis coefficient is less than 3, and higher than 3 for leptokurtic distribution (negative kurtosis). Investors will prefer positive kurtosis because of shorter tails which indicate fewer extreme outcomes are possible. This has positive benefits for portfolio construction as it makes final outcome more predictable.

One of the first empirical evidences on the investor's preference for positive skewness was presented by Kraus and Litzenberger (1976), who showed that in the case of higher skewness investors will accept lower expected returns. Mitton and Vorkink (2007) described the trade-off between diversification and skewness. They said that diversification limits the upside potential, which motivates investors to hold imperfectly diversified portfolios that have much higher positive skewness in order to increase their probability for higher payoffs. On the other hand, aversion to kurtosis (higher probability in the tails of the distribution) has been confirmed in a study by Dittmar (2002).

Following the evidence that investors are prepared to accept lower return and higher volatility in exchange for positive skewness and lower kurtosis (Boyer, Mitton & Vorkink, 2010) and the insight from Chen, Chen and Chen (2009) about the relationship between expected returns and downside risk, Amenc, Goltz, Martellini and Retkowsky (2010) proposed to use an alternative measure for expected returns that captured also higher order moments of returns. They used stock's semi-deviation as a proxy for expected return as it should also indirectly control for skewness and kurtosis. This is an example of indirect consideration of higher order moments.

On the other hand, introduction of estimators for higher order moments and comoments increase the dimensionality problem. But as Martellini and Ziemann (2010) demonstrated, it is possible to extend different statistical techniques from variance-covariance matrix estimation to higher order moments. They tested factor-based estimators, shrinkage estimators and constant correlation estimators and reported improved out of sample expected utility when using shrinkage or structured estimators instead of the sample estimators for coskewness and cokurtosis matrix. However, when it is still unclear which inputs for the mean-variance optimization are the best for the construction of efficient portfolios, additional research has to be done to improve higher order moments and comoments estimates before they can be widely used in portfolio optimizations.

# 4 EMPIRICAL ANALYSIS: EFFICIENT PORTFOLIO DIVERSIFICATION OF THE EUROPEAN LARGE AND SMALL-CAP STOCKS

In the last part of the thesis, I construct and test 9 different portfolios. First, research methodology will be described – selected portfolios, presentation of data, the construction of different portfolios and portfolio performance metrics. After that, the results of different portfolios on the sample of large and small-cap European stocks will be presented. Portfolios are evaluated based on the different performance metrics, such as risk and return, Sharpe ratio, tracking error, turnover, concentration and extreme risk. Additionally, I introduce brief sub-period analysis. Finally, the results of portfolio diversification of small and large-cap stocks will be compared to find their similarities and differences.

## 4.1 Research methodology

This section describes selected portfolios and reasons for their selection, stock returns that are used in the analysis, construction of different portfolios together with the estimates that were used and portfolio performance metrics.

### 4.1.1 Selected portfolios

Based on the reviewed literature, I decided to test the following approaches to portfolio diversification:

1. Market-cap weighted portfolio,
2. Equally weighted portfolio,
3. Fundamentally weighted portfolio with book value as a weight metric,
4. Minimum variance portfolio, and
5. Maximum Sharpe ratio portfolio.

I decided to exclude equally weighted risk contributions portfolio and the most diversified portfolio from the analysis because of the lack of the theoretical and empirical support as the appropriate stock portfolio weighting approaches. The ERC portfolio or risk parity was originally developed as an asset class weighting approach not stock portfolio weighting approach, and it has real usefulness in that field. Particularly, it helps managers in financial institutions to equally allocate risk among asset classes, in contrast to equally allocating capital. On the other hand, the MD portfolio's performance is questionable as it outperformed all other approaches in the study by Choueifaty, Froidure and Reynier (2013) for the US stock market in the period from 1992 to 2008. But when it was tested by Clarke, De Silva and Thorley (2013) for the US stock market from 1968 to 2012, its performance was the worst of all approaches, even worse than the market-cap portfolio. Hence, I will stick with only one risk-based approach - the minimum variance portfolio, which has strong theoretical basis, was widely examined and offered very good out-of-sample performance in all of the reviewed studies.

Other approaches do not need special explanation, other than fundamentally weighted portfolio where I chose to use only one fundamental metric, book value of equity. The book value of equity performed well in the study from Arnott, Hsu and Moore (2005), who proposed fundamental weighting. Therefore, I consider it the representative weighting metric for this approach. The rest of the approaches are market-cap weighted portfolio which will serve as a benchmark for all other approaches, equally weighted portfolio which has been documented as a simple way of improving market-cap weighted portfolio performance and maximum Sharpe ratio portfolio as a truly optimal portfolio.

Additionally, minimum variance portfolio and maximum Sharpe ratio portfolio will be estimated three times, each time with differently estimated portfolio moments which serve as the inputs to portfolio optimization. These portfolios are:

1. Minimum variance portfolio based on the sample var-cov matrix,
2. Minimum variance portfolio with the PCA used as an estimator of var-cov matrix,
3. Minimum variance portfolio with the shrinkage estimator for var-cov matrix,
4. Maximum Sharpe ratio portfolio based on the sample mean and sample var-cov matrix,
5. Maximum Sharpe ratio portfolio with the semi-deviation used as an estimator for stock expected returns and the PCA as an estimator of var-cov matrix,
6. Maximum Sharpe ratio portfolio with the semi-deviation used as an estimator for stock expected returns and the shrinkage estimator for var-cov matrix.

First, the MV portfolio and the MSR portfolio will be estimated with sample covariance matrix, and sample mean and sample covariance matrix, respectively. This is the simple Markowitz approach, but as I discussed in the third part of the thesis, sample estimates were shown to be bad estimates for numerous reasons. That is why I decided also to test the MV portfolio with the PCA and the shrinkage estimators, which were proven as a more reliable in estimating covariance matrix. When using principal component analysis, I decided to use 3 factors to explain the co-movement in stock returns as this was sufficient to capture the co-movement in returns. On the other hand, the shrinkage estimators are based on the paper of Ledoit and Wolf (2004) where covariance matrix is shrunk towards the constant correlation model.

The MSR portfolio with improved estimators for stock expected returns and covariance matrix uses the same PCA method and shrinkage estimators for estimating covariance matrix as the MV portfolios. The stock expected returns are estimated based on the stock's semi-deviation. This is the approach used in the paper from Amenc, Goltz, Lodh and Martellini (2010), where they achieved superior performance of the MSR portfolio when using semi-deviation as the proxy for the stock expected returns and the PCA analysis for estimating covariance matrix.

Semi-deviation is a robust measure of expected returns as it is consistent with the finance literature that higher risk should reflect in higher returns. Semi-deviation as a measure which takes into account only deviations below mean is thus more appropriate than a simple standard deviation which describes volatility above and below mean stock return. To be consistent with the approach in Amenc, Goltz, Lodh and Martellini (2010), I sorted stocks into their deciles based on their semi-deviation and then assign median value of each decile to all stocks in that decile. The last MSR portfolio is a combination of semi-deviation used for an expected return estimates and shrinkage estimators from the paper of Ledoit and Wolf (2004) for covariance matrix.

In total, the empirical analysis is performed on the 9 portfolios, i.e. the market-cap weighted portfolio, equally weighted portfolio, fundamentally weighted portfolio and 3 minimum variance and 3 maximum Sharpe ratio portfolios.

### 4.1.2 Data

The subject of the analysis are the European stocks with large and small market capitalization between January 1, 2002 and December 31, 2018 – that is a 17-year period. I wanted to compare how different portfolios that were described above performed on the sample of large and small-cap European stocks. Therefore, I chose STOXX Europe 600 index as a starting point for the analysis as it represents small, mid and large market capitalization companies from 17 European countries: Austria, Belgium, Denmark, Finland, France, Germany, Ireland, Italy, Luxembourg, the Netherlands, Norway, Poland, Portugal, Spain, Sweden, Switzerland and the United Kingdom (Stoxx, n.d.).

When I decided on the index from which to draw the sample, I downloaded constituent list from Bloomberg (2019) as of December 31, 2001 (the first available date for constituents was December 24, 2001). Then I ranked the stocks by their market capitalization and selected 50 stocks with the largest market capitalization and 50 stocks with the lowest market capitalization in the index. The requirement for stock selection was that data for their market capitalization and monthly return data were available throughout the whole period of the analysis. Selected stocks with their Bloomberg ticker and respective market capitalizations as of December 31, 2001 are presented in Appendix 2 for the large-cap stocks and Appendix 3 for the small-cap stocks. The small-cap stocks mostly satisfy the usual definition of small-cap stock, which is market capitalization between $300 million and $2 billion. Namely, the biggest stock in the small-cap sample had market capitalization of €1,93 billion and the smallest stock in the small-cap sample had the market-cap of €727 million.

In order to calculate different portfolios, I have downloaded market capitalization at the end of each year, as the portfolios were rebalanced yearly. At the same time, I have acquired book value of equity of all stocks on December 31 of each year, which I needed for the construction of fundamentally weighted portfolio (weighted by book value of equity). Next, the returns were downloaded on the monthly basis from January 1, 1997 till December 31, 2018. Additional 60 months (5 years) of returns is needed for optimized portfolios. Downloaded returns are gross returns with reinvestment of dividends. This is important as some companies pay dividends while others do not as they directly reinvest their profits to achieve higher growth in the future. The last component is a risk-free rate which is required for calculating Sharpe ratio. Appropriate risk-free rate for the European stocks is the return on the German government bond as it has the highest credit rating. Thus, I downloaded returns of a 1-year German government bond as it had the shortest maturity. Yearly returns were than recalculated on the monthly basis for each month. All of the data, from constituent list to market-cap, book value of equity, stock returns and risk-free returns were downloaded from Bloomberg (2019).

### 4.1.3 Stock portfolios construction

In order to construct and analyse different portfolios, I used RStudio (2016) together with the relevant R packages.[1] In the next two subchapters I present how simple portfolios (market-cap weighted portfolio, equally weighted portfolio and fundamentally weighted portfolio) and optimized portfolios (minimum variance portfolios and maximum Sharpe ratio portfolios) were constructed.

#### *4.1.3.1 Simple portfolios*

As already mentioned, simple portfolios include market-cap weighted portfolio (MCW portfolio), equally weighted portfolio (EW portfolio) and fundamentally weighted portfolio (FW portfolio).

In order to calculate simple portfolios, I first have to calculate portfolio weights. Market-cap weighted portfolio weights are calculated as it was shown in Equation (10). I divide market capitalization of each stock with the total market capitalization of all stocks in the portfolio. Consequently, weights have to sum to one. The analysis starts in 2002, so my first vector of weights is calculated based on the stock's market-cap on the December 31, 2001. The portfolios are rebalanced each year, so the new vector of weights is calculated each year at the end of the year. In total, there are 17 weights vectors. After I determined the weights, they are used in the combination with monthly returns and calculate portfolio return and other performance and risk measures.

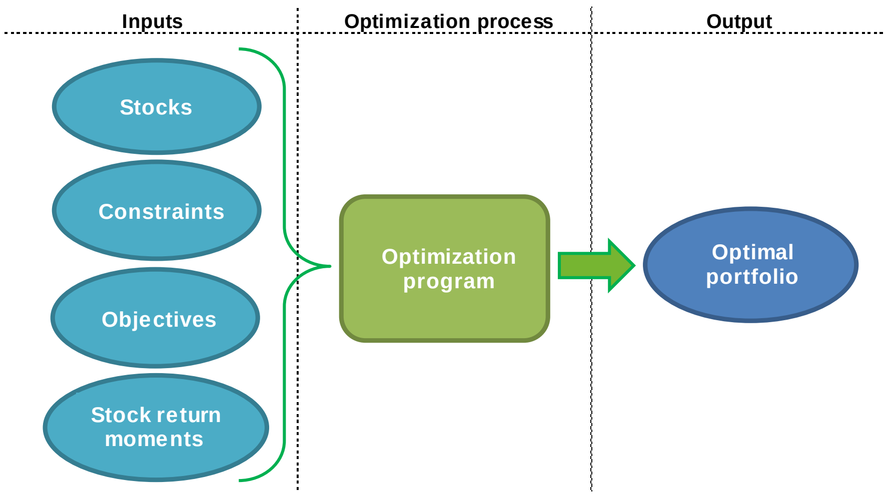
[1] To manipulate time series data, I used the "zoo" package (Zeileis & Grothendieck, 2005) and "xts" package (Ryan & Ulrich, 2018), the "PerformanceAnalytics" package (Peterson & Carl, 2018a) was used for performance and risk analysis, the "PortfolioAnalytics" package (Peterson & Carl, 2018b) was utilized for optimizing portfolios together with the solver packages "ROI" (Hornik, Meyer, Schwendinger & Theussl, 2019) and "DEoptim" (Ardia, Mullen, Peterson & Ulrich, 2016), and the "PeerPerformance" package (Ardia & Boudt, 2018) was used for Sharp ratio statistical tests. Finally, the "xlsx" package (Dragulescu & Arendt, 2018) was applied to export results to Excel.

Calculation of equally weighted portfolio proceeds in the same manner. Weights are determined with Equation (11). In this case each stock will have 2% weight in the portfolio. Again, I follow the same rebalancing procedure as for the market-cap weighted portfolio and rebalance weights back to 2% at the end of each year. Fundamentally weighted portfolio on the other hand is similar to market-cap weighted portfolio, with respect that weights are not market value of equity but book value of equity. Hence, the equation for calculating book value weights is the same as Equation (10), only the book value of equity is used instead of market capitalization.

#### *4.1.3.2 Optimized portfolios*

Calculation of optimized portfolios (minimum variance portfolios and maximum Sharpe ratio portfolios) is more complex than the construction of simple portfolios. When calculating simple portfolios, weights are already determined by market capitalization, book value of equity or they are set to be equal. In contrast, optimized portfolio weights are determined by optimization program which requires to specify portfolio assets, determine portfolio constraints and objectives, and estimate stock return moments. These inputs are then fed into optimization program which calculates optimal weights and produces the optimal portfolio as an output. Portfolio optimization workflow is presented in Figure 8.

Figure 8: Portfolio optimization workflow




*Source: Bennett (2015).*

The first input is stocks – in this case samples of large and small-cap stocks. Next, portfolio constraints have to be defined. In all optimized portfolio that I have calculated, constraints are the same: full investment and no short sales with maximum position limit of 20%. Full investment means weights have to sum to 1. Short sales are not allowed because most of portfolio managers and funds are long only and this also ensures better comparison with simple portfolios. Maximum position limit is useful because optimizers tend to concentrate portfolios in only few assets, and in order to prevent that I used 20% position limit to prevent too huge exposure to one stock. Minimum weight constraints were not imposed. The types of constraints which I decided to apply were also widely used in the empirical studies which I have reviewed in the second chapter of the thesis.

The third input into portfolio optimization is portfolio objective which is different for the MV portfolios and the MSR portfolios. In the "PortfolioAnalytics" package in RStudio, I simply set the objective function in the optimizer to find weights which minimize portfolio variance for the MV portfolio. In matrix notation, this can be written as (Clarke, De Silva & Thorley, 2013):

$$w^{MV} = \arg\min_{w} \ w^T * \sum * w \qquad (14)$$

Where $\sum$ is the covariance matrix and $w^{MV}$ is the vector of optimal weights that minimize portfolio variance.

On the other hand, the objective function for the MSR portfolio in the "PortfolioAnalytics" package is set to maximize Sharpe ratio, which is defined as excess expected return over risk-free rate divided with standard deviation. In matrix notation, this can be expressed as (Amenc, Goltz, Martellini & Retkowsky, 2010):

$$w^{MSR} = \arg\max_{w} \ \frac{w^T * \mu}{\sqrt{w^T * \sum * w}} \qquad (15)$$

Where $\sum$ is the covariance matrix, $\mu$ is the vector of excess expected returns over risk-free rate and $w^{MSR}$ is the vector of optimal weights that maximize expected returns per unit of risk.

The last input into portfolio optimization is stock return moments. Again, these differ among the MV portfolios and the MSR portfolios. The MV portfolios require only estimate of variance-covariance matrix which can be estimated with numerous methods. As I already explained in chapter 4.1.1, minimum variance portfolios that I have tested are based on the sample variance-covariance matrix, variance-covariance matrix that is estimated with the PCA with 3 factors, which imposes more structure, and shrinkage estimators, which shrink sample covariance matrix towards the constant correlation model.

At the same time the MSR portfolios also require expected return estimates besides estimates for covariance matrix. The estimates used, are sample mean and sample variance-covariance matrix, and semi-deviation used as an estimator for stock expected returns in combination with the PCA and shrinkage estimators for variance-covariance matrix. Explanation of selected methods together with more detailed description on how semi-deviation is used as stock expected returns is given in chapter 4.1.1.

Additional thing that has to be specified before the run of the optimization program is the length of estimation period. In this case, I use monthly stock returns from January 1, 1997 till December 31, 2018. More data on stock returns is required for optimized portfolios because of the training period. That is the period in which expected returns and covariance matrix are estimated and then used for calculation of optimized weights. In this case, I have used 60 months (5 years) of stock returns as a training period. Portfolios were rebalanced yearly, so I have applied the same rolling window of 60 months also for the subsequent years. The practice of using 5 years of data as the estimation period is quite common in the reviewed papers and also in other financial calculations as there is sufficient history but still allow for newer observations to influence the estimates.

After specifying everything from portfolio of stocks, constraints, objectives, estimates of stock return moments, rebalancing dates, training period and rolling window for estimation, the optimization program is run which uses specific solvers to search for optimal weights which give us the optimal portfolio. Because the results are out-of-sample, it is not necessarily that the MV portfolios will in fact achieve the lowest possible variance and the MSR portfolios to have the highest Sharpe ratio. That is why, I constructed portfolios with improved moment estimates as they should return better out-of-sample results.

### 4.1.4 Portfolio performance metrics

After the creation of all the portfolios, it is important to analyse them from different perspectives. Portfolio performance cannot be captured in only one number, so I calculated different performance metrics from simple annualized returns and Sharpe ratios to extreme risk measures. I will present all relevant metrics in this subchapter before I turn to the actual results.

First, I calculated cumulative returns which measure total portfolio gains in the analysed period from 2002 to 2018. Next, I report annualized average return calculated as geometric mean, which is more common in finance than arithmetic mean, because the returns are compounded. Annualized standard deviation is another basic metric that describes portfolio volatility below and above mean and is considered the most basic risk measure. Furthermore, excess portfolio return above the benchmark portfolio can be calculated, which is usually market-cap weighted portfolio or equally weighted portfolio.

Sharpe ratio is the most important ratio as it measures the risk adjusted return. Its calculation was presented in Equation (7). Another similar ratio is Sortino ratio which is calculated as the average excess return above minimum acceptable return (in my case I took risk-free rate) divided with downside deviation. Thus, Sortino ratio substitutes standard deviation with downside deviation which measures only deviations of negative portfolio returns (Kenton, 2019a). The third similar metric is Treynor ratio which is computed as the excess portfolio return above risk-free rate divided with portfolio's beta. Again, the difference is only in the measurement of risk, which is now represented with beta as a measure for systematic risk (Kenton, 2019b).

Important metric for portfolio managers is tracking error which measures the dispersion of returns between a manager's portfolio and benchmark portfolio (Kassam, Gupta, Jain, Kouzmenko & Briand, 2013). This is important as portfolio managers are usually measured against a market-cap weighted benchmark and their portfolio may beat the cap-weighted portfolio on the long run but underperform in the short-run. Hence, they are concerned with tracking error of their portfolio relative to the benchmark because underperformance can result in the termination of their contract. This problem was researched by Amenc, Goltz, Lodh and Martellini (2012) who proposed that managers reduce the tracking error and limit the risk of underperformance by combining the MV portfolio and the MSR portfolio which have different performance characteristics. Alternatively, they can implement relative risk control for separate optimized portfolio (or their combination) what is even better in reducing extreme tracking error. Thus, it is important for portfolio managers to look at the tracking error of different portfolio strategies.

Information ratio is calculated as an active return divided with tracking error, where the active return is the difference between portfolio return and benchmark return (Kassam, Gupta, Jain, Kouzmenko & Briand, 2013). Portfolio managers who achieve higher information ratio are thus capable of generating higher returns while limiting the tracking error to benchmark portfolio.

Another important metric is portfolio turnover, which measures the change in the composition of a portfolio at each rebalancing date. Two-way turnover aggregates both, weight increases and weight decreases. One-way turnover is thus simply the half of two-way turnover (Kassam, Gupta, Jain, Kouzmenko & Briand, 2013). One-way turnover is important because transaction costs were not included into the analysis which is common practice as transaction costs are different for large and small-cap stocks. Optimized portfolios normally have much higher turnovers than simple portfolios, thus, it is important to look at them. Based on the annual turnover I calculated indifference transaction costs which measure, when the transaction costs would offset the excess return of a portfolio over the benchmark.

Measurement of portfolio concentration is also important. I already explained that optimized portfolios tend to allocate large weights to a few stocks, which makes them relatively imbalanced. The measure for portfolio concentration is effective number of stocks, which ranges from 1 (everything is invested in one stock) to the number of stocks in a portfolio (for an equally weighted portfolio) (Kassam, Gupta, Jain, Kouzmenko & Briand, 2013). The higher number is better as it indicates reasonably diversified portfolio.

Different measures of extreme portfolio risk include annual semi-deviation that measures only deviations below the mean (Amenc, Goltz, Martellini & Retkowsky, 2010). Next is Value at Risk (hereinafter: VaR) where the maximum possible loss has been measured for a given time horizon and confidence interval (usually 95% or 99%). Based on the historical observations of portfolio returns, 95% VaR for one-month horizon can thus be calculated, which can return 7%. The explanation is that with 95% confidence, it is expected for the portfolio loss in the next month not to exceed 7% (Kassam, Gupta, Jain, Kouzmenko & Briand, 2013).

Similarly, one can calculate Expected Shortfall (hereinafter: ES) which measures for a given time horizon and confidence interval (usually 95% or 99%) the expected loss. Again, one can calculate ES from historical observations of portfolio returns for one month at 95% confidence level and obtain 10%. Here, the explanation is: if the loss exceeds the 5% percent of the worst losses, then one can expect the average loss of 10% (Kassam, Gupta, Jain, Kouzmenko & Briand, 2013).

The most extreme measure of risk is the maximum drawdown which measures the percentage drop from the peak to trough in the observed period. At the same time, one can calculate maximum drawdown period as the number of days/months/years it takes to recoup the losses (Kassam, Gupta, Jain, Kouzmenko & Briand, 2013). Additionally, one can report the skewness and kurtosis of portfolio returns distribution.

In order to see if the differences in average return, standard deviation, and Sharpe ratio between the market-cap weighted portfolio and all the other portfolios are statistically significant, appropriate statistical tests can be performed. I have used the same tests as Amenc, Goltz, Martellini and Retkowsky (2010) and Amenc, Goltz, Lodh and Martellini (2012), i.e. the paired two-sided t-test for the average returns, the F-test for volatility and bootstrap approach for the Sharpe ratio. The differences are computed from annualized values and geometric average is used for average returns. I report the p-values for differences, where I state that difference is statistically significant if the p-value is lower than 0.05 or 5%.

The last thing I have included in the analysis is the CAPM analysis. Here, I have calculated regressions of monthly returns of the examined portfolios on the market-cap weighted portfolio. I have reported the annualized alpha, beta and R-squared. The beta from the CAPM analysis shows if a portfolio moves more, equal or less than the market-cap portfolio. The significant alpha indicates returns that are higher than the market portfolio (Amenc, Goltz, Martellini and Retkowsky, 2010). Coefficient of determination or R-squared tells the percentage of explained variability in the dependent variable from the independent variable(s).

## 4.2 Efficient portfolio diversification of the European large-cap stocks

Now that I have explained the research methodology, I present the results of the empirical analysis. First, I present the results for the sample of the European large-cap stocks. Table 2 in panel A shows performance statistics and in panel B differences in returns, volatilities and Sharpe ratios are tested for statistical significance. Portfolios shown in the table are market-cap weighted portfolio (MCW portfolio), equally weighted portfolio (EW portfolio), fundamentally weighted portfolio with book value of equity (FW portfolio), minimum variance portfolios (MV portfolios) with differently estimated covariance matrix (method is denoted in parentheses) and maximum Sharpe ratio portfolios (MSR portfolios) with differently estimated expected returns and covariance matrix.

Table 2: Performance statistics and difference over the market-cap weighted portfolio of the European large-cap stock portfolios in the period from 2002 till 2018

| | MCW portfolio | EW portfolio | FW portfolio | MV portfolio (Sample) | MV portfolio (PCA) | MV portfolio (Shrink) | MSR portfolio (Sample) | MSR portfolio (SemiDev, PCA) | MSR portfolio (SemiDev, Shrink) |
|---|---|---|---|---|---|---|---|---|---|
| *Panel A: Perfromance Statistics* | | | | | | | | | |
| Cummulative Return | 84.69% | 106.69% | 55.78% | 191.18% | 166.87% | 196.96% | 97.11% | 190.01% | 181.13% |
| Ann. average return (geometric) | 3.67% | 4.36% | 2.64% | 6.49% | 5.94% | 6.61% | 4.07% | 6.46% | 6.27% |
| Ann. standard deviation | 14.07% | 16.56% | 17.65% | 11.51% | 11.23% | 11.31% | 13.27% | 12.23% | 12.25% |
| Sharpe ratio | 0.17 | 0.18 | 0.08 | 0.45 | 0.41 | 0.47 | 0.21 | 0.42 | 0.40 |
| Sortino ratio | 0.06 | 0.08 | 0.04 | 0.17 | 0.15 | 0.17 | 0.07 | 0.15 | 0.15 |
| Treynor ratio | 0.02 | 0.03 | 0.01 | 0.07 | 0.07 | 0.08 | 0.03 | 0.06 | 0.06 |
| Tracking error | 0.00% | 4.31% | 1.79% | 9.73% | 7.47% | 10.11% | 1.13% | 9.99% | 9.24% |
| Information ratio | - | 0.16 | -0.58 | 0.29 | 0.30 | 0.29 | 0.35 | 0.28 | 0.28 |
| Ann. one-way turnover | 7.04% | 10.61% | 10.95% | 31.42% | 21.08% | 22.63% | 43.85% | 51.84% | 48.38% |
| Indiff. transaction costs | - | 19.27% | 0.00% | 11.54% | 16.16% | 18.84% | 1.08% | 6.22% | 6.28% |
| Effective N | 33.43 | 50.00 | 27.01 | 8.39 | 9.18 | 8.23 | 7.11 | 11.60 | 11.79 |
| *Panel B: Difference over the MCW portfolio* | | | | | | | | | |
| Diff. in returns | - | 0.69% | -1.03% | 2.81% | 2.27% | 2.94% | 0.40% | 2.79% | 2.59% |
| P-value | - | 0.125 | 0.709 | 0.263 | 0.400 | 0.213 | 0.956 | 0.155 | 0.174 |
| Diff. in volatility | - | 2.49% | 3.58% | -2.56% | -2.84% | -2.76% | -0.80% | -1.84% | -1.82% |
| P-value | - | 0.642 | 0.500 | 0.216 | 0.231 | 0.217 | 0.703 | 0.434 | 0.569 |
| Diff. in Sharpe ratio | - | 0.02 | -0.09 | **0.28** | **0.24** | **0.30** | 0.04 | **0.25** | **0.23** |
| P-value | - | 0.417 | 0.094 | 0.010 | 0.043 | 0.009 | 0.723 | 0.012 | 0.046 |

*Source: Own work.*

From the above table, one can see that optimized portfolios, namely, three MV portfolios and the MSR portfolios with advanced moment estimates achieved the highest cumulative returns in the observed period from January 1, 2002 to December 31, 2018. For the simple portfolios, one can see that EW portfolio achieved higher average return and higher standard deviation than the MCW portfolio which is consistent with the findings of other authors. On the other hand, the FW portfolio performed worse in both terms and poses question of suitability of this approach. Furthermore, one can see that annualized standard deviation is the lowest for three MV portfolios, with little difference between methods for estimating covariance matrix. This indicates that the curse of dimensionality in the estimation process of the covariance matrix is not present for portfolio of only 50 stocks. Annualized average geometric returns are the highest for optimized portfolios, expect for the MSR portfolio estimated with sample moments which has lower average return than the EW portfolio.

Next, I review the Sharpe ratio, Sortino ratio and Treynor ratio of different portfolios. Again, optimized portfolios (except for the MSR portfolio with sample estimates) achieved Sharpe ratios that are more than 100% higher than those of the simple portfolios. Sharpe ratios of two MV portfolios are also higher than the best performing MSR portfolio which confirms the findings from previous studies (e.g. Amenc, Goltz, Lodh & Martellini, 2012) that out-of-sample MV portfolios are very close to the MSR portfolios as they avoid estimation error in the expected returns. Sortino ratio and Treynor ratio confirm what can already be observed with Sharpe ratio.

Tracking error and information ratio is presented next. These metrics are important for portfolio managers who care about their performance with respect to a benchmark. To calculate tracking error and information ratio, I took the MCW portfolio as the benchmark. One can see that average tracking error is the highest for optimized portfolios, except for the MSR portfolio with sample moments. As it was explained in chapter 4.1.4, high tracking error can be problematic for portfolio managers. To minimize this problem, they can add tracking error as an optimization objective to reduce the chance of underperforming the benchmark. Despite the high tracking error, information ratio of optimized portfolios is higher than those of the simple portfolios.

Annualized one-way turnover is also important as it indicates the average percentage of stocks that have to be substituted in the portfolio at each rebalancing date. Optimized portfolios have much higher turnover than simple portfolios which is normal but this reduces the chance of outperforming the benchmark when one includes transaction costs. That is why I have included indifference transaction costs in the table. One can see that the MV portfolios are more attractive than the MSR portfolios from these two perspectives. However, it is unlikely that any of the optimized portfolios (except for the MSR portfolio with sample estimates) will have transaction costs that high to eliminate the advantage over the MCW portfolio. Additionally, if portfolio manager is concerned with high turnover, he can use optimal turnover control in the optimization program where he limits the allowed annual rebalancing in a same way that he can specify maximum allowed tracking error. However, additional portfolio objectives and constraints usually come at the cost of lower portfolio efficiency (Sharpe ratio).

Effective N is a measure of portfolio concentration and can take values between 1 (investment in only one stock) and the number of all stocks (N) in a portoflio (for an equally weighted portfolio). One can see that optimized portfolios are poorly diversified as they have around 3 times less stocks in the portfolio than simple portfolios. This is normal for unconstrained portfolio optimizations and is their inherent characteristic as they usually grab only a small number of stocks, especially minimum variance portfolios. This problem can be mitigated by imposing stricter lower and upper bound constraints on portfolio weights to enhance diversification. However, additional constraints can reduce the portfolio efficiency.

In panel B, differences over the MCW portfolio were calculated and tested with appropriate statistical tests as explained in chapter 4.1.4. For the average returns, I have used the paired two-sided t-test, for the volatility the F-test and bootstrap approach for the Sharpe ratio. The differences are computed from annualized values and geometric average was used for average returns. Differences where P-values are lower than 0.05 are indicated in bold. One can see that differences in the returns and the volatility are not statistically significant, although some portfolios are closer to zero than others (the MSR portfolios for average returns and the MV portfolios for volatility). The power of the tests could be improved if I had included longer observation period, which was not possible. Standard error which affects the calculation can also be problematic as the examined period had quite a few turbulent years. Nevertheless, Sharpe ratios are statistically significant for all three MV portfolios and two MSR portfolios which indicate the higher efficiency of optimized portfolios.

Next, I have to examine portfolios from the perspective of extreme risks which are presented in Table 3. The calculated metrics were described in chapter 4.1.4, hence, I will only comment the results.

Table 3: Extreme risk measures of the European large-cap stock portfolios in the period from 2002 till 2018

| | MCW portfolio | EW portfolio | FW portfolio | MV portfolio (Sample) | MV portfolio (PCA) | MV portfolio (Shrink) | MSR portfolio (Sample) | MSR portfolio (SemiDev, PCA) | MSR portfolio (SemiDev, Shrink) |
|---|---|---|---|---|---|---|---|---|---|
| Ann. semi deviation | 10.64% | 12.24% | 13.03% | 8.57% | 8.44% | 8.40% | 9.99% | 9.22% | 9.10% |
| 95% Value at Risk | -6.77% | -7.56% | -8.22% | -5.19% | -5.17% | -5.12% | -6.38% | -5.65% | -5.58% |
| 95% Expected Shortfall | -9.94% | -11.58% | -12.50% | -7.88% | -7.64% | -7.44% | -9.09% | -7.87% | -7.94% |
| Maximum drawdown | -44.92% | -48.98% | -53.76% | -29.72% | -30.94% | -31.06% | -42.08% | -34.01% | -36.16% |
| Maximum drawdown period (months) | 69 | 71 | 76 | 34 | 42 | 39 | 63 | 39 | 63 |
| Skewness | -0.54 | -0.28 | -0.28 | -0.46 | -0.53 | -0.46 | -0.55 | -0.51 | -0.41 |
| Kurtosis | 4.27 | 4.90 | 4.89 | 4.43 | 4.20 | 4.00 | 3.97 | 3.65 | 3.83 |

*Source: Own work.*

In Table 3, one can see that annualized semi-deviation, VaR, ES and maximum drawdown are all much lower for optimized portfolios. This confirms that simple portfolios are not just less efficient but are also more exposed to extreme risks. The least improvement in optimized portfolios is seen in the MSR portfolio with sample moments which again confirms that past mean returns are bad estimates for future returns. If the goal of an investor is minimization of risk, then the MV portfolios should be selected as they are the best for minimizing risk. They beat the MSR portfolios in all aspects of extreme risks. Maximum drawdown period is also an interesting metric as it shows how many months it took for portfolio to recoup the biggest loss. Here, the MV portfolios also performed the best which confirms their superiority in the preservation of capital. Additionally, I report skewness and kurtosis of portfolio return distributions. One can see that all portfolios are negatively skewed (higher probability of large negative returns than positive) and have higher kurtosis (fatter tails) than normal distribution (higher than 3). Interestingly, negative skewness is the lowest for the EW and the FW portfolios, while kurtosis is the most similar to normal distribution for the MSR portfolios.

Table 4 presents the CAPM analysis of the portfolios where the MCW portfolio is taken as the benchmark. I calculated annualized alpha and beta from single factor analysis with respective P-values. The percentage of explained variability is also shown in the last row.

Table 4: CAPM analysis of the European large-cap stock portfolios in the period from 2002 till 2018

| | EW portfolio | FW portfolio | MV portfolio (Sample) | MV portfolio (PCA) | MV portfolio (Shrink) | MSR portfolio (Sample) | MSR portfolio (SemiDev, PCA) | MSR portfolio (SemiDev, Shrink) |
|---|---|---|---|---|---|---|---|---|
| Ann. Alpha | 0.34% | -1.37% | **3.74%** | **3.40%** | **3.98%** | 1.21% | **3.35%** | **3.10%** |
| P-value | 0.680 | 0.261 | 0.011 | 0.031 | 0.011 | 0.486 | 0.005 | 0.005 |
| Beta | **1.15** | **1.20** | **0.70** | **0.65** | **0.67** | **0.80** | **0.80** | **0.81** |
| P-value | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| R-squared | 95.9% | 92.0% | 73.1% | 67.3% | 68.9% | 71.4% | 84.3% | 86.8% |

*Source: Own work.*

In the above table, one can see that annualized alpha is significant for three MV portfolios and two MSR portfolios which indicate that these portfolios significantly outperformed the MCW benchmark. On the other hand, betas show that the EW portfolio and the FW portfolio move more than the market portfolio, while the MV and the MSR portfolios move much less. Similar findings about the MV portfolios sensitivity to the market were reported in studies from Clarke, De Silva and Thorley (2006; 2013) who reported low betas for unconstrained portfolios and lower R-squared. By imposing constraints portfolios moved closer with the market (higher beta) which also increased R-squared but decreased alpha to some extent. R-squared around 70% for the MV portfolios thus indicates that additional factors could explain higher returns of the optimized portfolios.

The last thing I have calculated are risk and return together with Sharpe ratio for two sub-periods, from January 1, 2002 till December 31, 2009 and from January 1, 2010 till December 31, 2018. I decided to split the analysed period to two sub-periods, where the first sub-period is 8 years long and presents the unstable economic and market environment, including two stock market downturns, first between years 2002 and 2003 and second when the financial crisis hit in 2008 till 2009. The second sub-period is 9 years long and it began after the stock market reached its bottom. These 9 years were characterized mainly by the bull markets with less volatility and downturns. Thus, it is interesting to analyse the performance differences in these two sub-periods. Results are presented in Table 5.

Table 5: Risk and return of the European large-cap stock portfolios in two sub-periods, from 2002 till 2009 and from 2010 till 2018

| | MCW portfolio | EW portfolio | FW portfolio | MV portfolio (Sample) | MV portfolio (PCA) | MV portfolio (Shrink) | MSR portfolio (Sample) | MSR portfolio (SemiDev, PCA) | MSR portfolio (SemiDev, Shrink) |
|---|---|---|---|---|---|---|---|---|---|
| 2002 - 2009 | | | | | | | | | |
| Ann. average return (geometric) | 0.93% | 1.77% | 0.23% | 2.80% | 3.08% | 3.02% | 3.03% | 3.65% | 2.44% |
| Ann. standard deviation | 16.42% | 19.64% | 20.17% | 12.61% | 12.51% | 12.29% | 15.96% | 13.42% | 13.34% |
| Sharpe ratio | -0.11 | -0.05 | -0.12 | 0.00 | 0.02 | 0.02 | 0.01 | 0.06 | -0.03 |
| Beta | 1.00 | 1.17 | 1.19 | 0.67 | 0.66 | 0.64 | 0.84 | 0.77 | 0.77 |
| 2010 - 2018 | | | | | | | | | |
| Ann. average return (geometric) | 6.18% | 6.72% | 4.83% | 9.88% | 8.56% | 9.91% | 5.01% | 9.03% | 9.79% |
| Ann. standard deviation | 11.63% | 13.31% | 15.14% | 10.42% | 9.96% | 10.34% | 10.39% | 11.10% | 11.18% |
| Sharpe ratio | 0.54 | 0.51 | 0.32 | 0.96 | 0.87 | 0.97 | 0.49 | 0.82 | 0.88 |
| Beta | 1.00 | 1.11 | 1.22 | 0.74 | 0.65 | 0.71 | 0.72 | 0.85 | 0.88 |

*Source: Own work.*

In the above table, one can see the dismal performance of equity portfolios in the first sub-period with very low annualized average returns and high standard deviations. Consequently, Sharp ratios are negative for simple portfolios and for the last MSR portfolio while other optimized portfolios have barely positive Sharpe ratios. However, optimized portfolios, especially the MV portfolios, were successful in minimizing volatility compared to the simple portfolios. They also rewarded investors with a little bit higher average return which resulted in slightly positive Sharpe ratios. On the other hand, situation is completely different for the succeeding period, with much higher returns and Sharpe ratios and a lot lower volatility. Again, the MV portfolios had the lowest standard deviation, although the difference is not that noticeable. At the same time, the MV portfolios achieved high returns which resulted in the highest Sharpe ratios of all portfolios. Again, out of all the optimized portfolios the MSR portfolio with sample estimates performed the worst as it had Sharpe ratio lower than the MCW portfolio and the EW portfolio. Performance of other MSR portfolios was not bad but they registered lower Sharpe ratios than the MV portfolios. When taking a look at the beta, one cannot see any relationship as some portfolios had lower beta in the market downturn and higher beta in the bull market environment while others moved more closely with the market in the first sub-period and less in the second sub-period.

### 4.3 Efficient portfolio diversification of the European small-cap stocks

Now, I turn to the results for the sample of the European small-cap stocks. Portfolios from the small-cap stocks are constructed in the same way as for the large-cap stocks and are denoted in the same manner. Results are presented in Table 6, where panel A shows the performance statistics and panel B tests the differences in returns, volatilities and Sharpe ratios for statistical significance. First, one can observe that small-cap stock portfolios have much higher cumulative returns and annualized average returns than large-cap stock portfolios. Differences in standard deviations are mixed and depend on the chosen portfolio. The differences and similarities between portfolio diversification of large and small-cap stocks will be covered in detail in the next chapter, but now I will focus only on the results for small-cap stock portfolios.

Table 6: Performance statistics and difference over the market-cap weighted portfolio of the European small-cap stock portfolios in the period from 2002 till 2018

| | MCW portfolio | EW portfolio | FW portfolio | MV portfolio (Sample) | MV portfolio (PCA) | MV portfolio (Shrink) | MSR portfolio (Sample) | MSR portfolio (SemiDev, PCA) | MSR portfolio (SemiDev, Shrink) |
|---|---|---|---|---|---|---|---|---|---|
| *Panel A: Perfromance Statistics* | | | | | | | | | |
| Cummulative Return | 273.14% | 418.79% | 245.23% | 385.67% | 306.00% | 300.01% | 254.02% | 545.32% | 464.89% |
| Ann. average return (geometric) | 8.05% | 10.17% | 7.56% | 9.74% | 8.59% | 8.50% | 7.72% | 11.59% | 10.72% |
| Ann. standard deviation | 17.38% | 18.35% | 20.75% | 10.46% | 10.33% | 10.49% | 11.69% | 13.92% | 14.93% |
| Sharpe ratio | 0.39 | 0.48 | 0.30 | 0.80 | 0.70 | 0.68 | 0.55 | 0.73 | 0.63 |
| Sortino ratio | 0.16 | 0.21 | 0.14 | 0.29 | 0.25 | 0.25 | 0.19 | 0.29 | 0.25 |
| Treynor ratio | 0.07 | 0.09 | 0.05 | 0.19 | 0.17 | 0.16 | 0.13 | 0.14 | 0.12 |
| Tracking error | 0.00% | 8.61% | 0.62% | 2.35% | 2.10% | 2.41% | 4.79% | 11.09% | 8.41% |
| Information ratio | - | 0.25 | -0.79 | 0.72 | 0.26 | 0.18 | -0.07 | 0.32 | 0.32 |
| Ann. one-way turnover | 10.20% | 16.16% | 15.68% | 21.38% | 18.17% | 17.36% | 33.56% | 48.91% | 49.51% |
| Indiff. transaction costs | - | 35.53% | 0.00% | 15.10% | 6.75% | 6.19% | 0.00% | 9.14% | 6.79% |
| Effective N | 25.46 | 50.00 | 26.53 | 8.29 | 9.19 | 7.83 | 7.90 | 12.23 | 11.78 |
| *Panel B: Difference over the MCW portfolio* | | | | | | | | | |
| Diff. in returns | - | 2.12% | -0.49% | 1.69% | 0.54% | 0.44% | -0.33% | 3.54% | 2.67% |
| P-value | - | 0.118 | 0.587 | 0.828 | 0.605 | 0.621 | 0.448 | 0.392 | 0.494 |
| Diff. in volatility | - | 0.97% | 3.37% | **-6.92%** | **-7.05%** | **-6.90%** | -5.70% | -3.47% | -2.46% |
| P-value | - | 0.756 | 0.470 | 0.042 | 0.042 | 0.047 | 0.125 | 0.322 | 0.449 |
| Diff. in Sharpe ratio | - | 0.09 | -0.09 | **0.42** | 0.32 | 0.30 | 0.16 | **0.35** | **0.24** |
| P-value | - | 0.282 | 0.249 | 0.014 | 0.103 | 0.190 | 0.347 | 0.000 | 0.026 |

*Source: Own work.*

The cumulative return for the sample of the small-cap stocks is the highest for the MSR portfolio, where stocks' semi-deviation is used for the expected returns and the PCA for the covariance matrix. At the same time, this portfolio also had the highest annualized average return. In terms of cumulative returns and average returns performance of the EW portfolio, the MV portfolio with sample covariance matrix and the MSR portfolio with stock's semi-deviation used for the expected returns and the shrinkage estimators for covariance matrix, also showed a good performance. Turning to the measure of the volatility, one can see that all three MV portfolios were successful in minimizing the portfolio's standard deviation, again with minimal differences, which indicates that I do not have the problems with covariance matrix dimensionality in the case of portfolio with 50 stocks. The MSR portfolios also had lower standard deviations than the simple portfolios. Again, the worst performing portfolio in all aspects is the FW portfolio which definitely should not be used by any investor.

When taking a look at the Sharpe ratios, one can see that optimized portfolios achieved the highest Sharpe ratios. Surprisingly, the highest Sharpe ratio was realized by the MV portfolio with sample covariance matrix. This is due to the much higher average return than the other two MV portfolios. The MSR portfolio with semi-deviation and the PCA also performed quite well. Sortino ratio mostly follows the findings from Sharpe ratio analysis but it equalizes the MV portfolio (Sample) and the MSR portfolio (SemiDev, PCA) as the best performing portfolios. Treynor ratio, which substitutes standard deviation and semi-deviation in the denominator with portfolio's beta, is the highest for the MV portfolios as they are less volatile than the market (this will be seen in the CAPM analysis).

Turning to the tracking error, one can see that the MV portfolios have surprisingly low tracking error, while it is much higher for the MSR portfolios (expect for portfolio with sample estimates) and the EW portfolio. Information ratio is the highest for the MV portfolio with sample covariance matrix because of a quite big difference in returns and low tracking error. However, other portfolios' information ratios are also not bad (except for the FW portfolio and the MSR portfolio with sample estimates), but the MV portfolio with sample covariance matrix stands out by a big margin.

Next, one can see that annual turnover is the lowest for simple portfolios, followed by the MV portfolios which do not have much higher turnover. The worst in terms of turnover are the MSR portfolios with improved return moments, which have turnover close to 50%. Indifference transaction costs are the highest for the EW portfolio (35.5%), and they range between 6.2% and 15.1% for optimized portfolios. Thus, it is unlikely that transaction costs would erase the difference in returns. The effective numbers of stocks in the portfolio show the same dynamic as for the large-cap stock portfolios, namely, optimized portfolios are poorly diversified and would need additional minimum and maximum weight constraints to improve the diversification.

Tests for statistical significance reject the difference in returns over the MCW portfolio for all the portfolios, but they confirm that lower volatility of all the MV portfolios is statistically significant. Test for statistical significance of Sharpe ratios confirms the difference only for the MV portfolio with sample covariance matrix and two MSR ratio portfolios with improved estimators for return moments. Surprisingly, the difference is not confirmed for other two MV portfolios, as it was in the case of the large-cap stock portfolios, although the difference in Sharpe ratios is more than 0.30. However, statistical tests confirmed that the MV portfolios are successful in minimizing variance, which is their main goal, and the MSR portfolios with improved estimators are successful in improving Sharpe ratio. Thus, optimized portfolios are successful in achieving their objectives.

Table 7 presents the extreme risk measures for the small-cap stock portfolios. Extreme risk measures are even more important for the small-cap stock portfolios because small stocks are much more volatile than the large-cap stocks.

Table 7: Extreme risk measures of the European small-cap stock portfolios in the period from 2002 till 2018

| | MCW portfolio | EW portfolio | FW portfolio | MV portfolio (Sample) | MV portfolio (PCA) | MV portfolio (Shrink) | MSR portfolio (Sample) | MSR portfolio (SemiDev, PCA) | MSR portfolio (SemiDev, Shrink) |
|---|---|---|---|---|---|---|---|---|---|
| Ann. semi deviation | 13.36% | 13.18% | 15.01% | 8.18% | 7.97% | 7.91% | 9.31% | 10.72% | 11.30% |
| 95% Value at Risk | -8.22% | -7.24% | -8.34% | -4.70% | -4.61% | -4.53% | -5.70% | -6.32% | -6.65% |
| 95% Expected Shortfall | -14.47% | -11.47% | -13.20% | -7.56% | -6.71% | -6.61% | -9.82% | -10.56% | -10.38% |
| Maximum drawdown | -65.35% | -61.57% | -71.70% | -44.93% | -45.40% | -45.02% | -49.42% | -50.04% | -51.99% |
| Maximum drawdown period (months) | 76 | 67 | 77 | 72 | 76 | 90 | 74 | 69 | 69 |
| Skewness | -0.80 | 0.02 | 0.01 | -0.85 | -0.62 | -0.41 | -1.22 | -0.86 | -0.57 |
| Kurtosis | 6.18 | 7.40 | 8.62 | 5.04 | 3.99 | 3.92 | 6.48 | 5.44 | 4.62 |

*Source: Own work.*

One can observe that all optimized portfolios achieved lower annualized semi-deviation, Value at Risk, Expected Shortfall and maximum drawdown than the simple portfolios. All three MV portfolios also stand apart from the MSR portfolios in those four extreme risk metrics. Thus, for everyone whose first concern is preservation of capital, the MV portfolios are an obvious choice. However, the MSR portfolios also should not be neglected as they are still much better than the simple portfolios, whose protection against extreme risks is really worrying. Interestingly, maximum drawdown period does not point to such big differences and shows that the EW portfolio needed the least time to recover the largest loss. Furthermore, one can see that all portfolios except the EW portfolio and the FW portfolio are negatively skewed. However, when looking at the kurtosis of portfolio return distributions, the EW portfolio and the FW portfolio performed the worst with much fatter tails than the normal distribution. In the case of kurtosis, optimized portfolios again show their superiority in comparison to simple portfolios, especially the MV portfolios with improved estimators which reduced kurtosis towards that of the normal distribution by considerable margin.

The CAPM analysis is presented next. I report the annualized alpha and beta with their significance levels, and explained variability. Results are presented in the table below.

Table 8: CAPM analysis of the European small-cap stock portfolios in the period from 2002 till 2018

| | EW portfolio | FW portfolio | MV portfolio (Sample) | MV portfolio (PCA) | MV portfolio (Shrink) | MSR portfolio (Sample) | MSR portfolio (SemiDev, PCA) | MSR portfolio (SemiDev, Shrink) |
|---|---|---|---|---|---|---|---|---|
| Ann. Alpha | 1.88% | -1.23% | **5.75%** | **4.81%** | **4.63%** | 3.65% | **5.19%** | **3.96%** |
| P-value | 0.100 | 0.378 | 0.001 | 0.007 | 0.009 | 0.069 | 0.000 | 0.005 |
| Beta | **1.02** | **1.15** | **0.44** | **0.43** | **0.44** | **0.48** | **0.73** | **0.79** |
| P-value | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| R-squared | 93.7% | 92.6% | 54.3% | 51.9% | 53.2% | 51.9% | 83.5% | 85.4% |

*Source: Own work.*

In Table 8, one can see statistically significant annualized alphas for three MV portfolios and two MSR portfolios. Significant alphas indicate outperformance of the MCW portfolio. Looking at the betas, one can see that the EW portfolio's beta is close to the market beta, while the FW portfolio is more volatile with beta of 1.15. On the other hand, the MV portfolios and the MSR portfolio with sample estimates have very low beta. As I explained already in the previous chapter, this is normal for the MV portfolios. The MSR portfolios with improved portfolio moments have beta which is lower than the market beta but much higher than those of the MV portfolios. R-squared is especially low for the MV portfolios which indicate that other factors could help explain the returns of these portfolios, and also the MSR portfolios. These additional risk factors could be the SMB and the HML factors proposed by Fama and French (1993), which could reveal the exposure of the MV portfolios to other factors than the market factor which are driving the MV portfolio returns.

The last thing is to look at how different portfolio diversification approaches behaved in two sub-periods. The same as for the large-cap stock portfolios sub-periods stretch from January 1, 2002 till December 31, 2009 and from January 1, 2010 till December 31, 2018. Annualized average returns, standard deviations and Sharpe ratios for two sub-periods are presented in the table below.

Table 9: Risk and return of the European small-cap stock portfolios in two sub-periods, from 2002 till 2009 and from 2010 till 2018

| | MCW portfolio | EW portfolio | FW portfolio | MV portfolio (Sample) | MV portfolio (PCA) | MV portfolio (Shrink) | MSR portfolio (Sample) | MSR portfolio (SemiDev, PCA) | MSR portfolio (SemiDev, Shrink) |
|---|---|---|---|---|---|---|---|---|---|
| 2002 - 2009 | | | | | | | | | |
| Ann. average return (geometric) | 6.71% | 10.36% | 7.78% | 9.16% | 9.11% | 9.67% | 5.09% | 10.38% | 10.48% |
| Ann. standard deviation | 20.93% | 22.29% | 25.41% | 12.27% | 12.10% | 12.12% | 13.52% | 16.71% | 17.14% |
| Sharpe ratio | 0.18 | 0.33 | 0.19 | 0.51 | 0.51 | 0.55 | 0.17 | 0.44 | 0.44 |
| Beta | 1.00 | 1.04 | 1.18 | 0.46 | 0.44 | 0.45 | 0.50 | 0.76 | 0.78 |
| 2010 - 2018 | | | | | | | | | |
| Ann. average return (geometric) | 9.27% | 10.00% | 7.37% | 10.27% | 8.13% | 7.47% | 10.11% | 12.68% | 10.94% |
| Ann. standard deviation | 13.58% | 14.07% | 15.60% | 8.59% | 8.52% | 8.84% | 9.80% | 10.93% | 12.72% |
| Sharpe ratio | 0.69 | 0.72 | 0.48 | 1.20 | 0.96 | 0.85 | 1.04 | 1.17 | 0.87 |
| Beta | 1.00 | 0.99 | 1.09 | 0.42 | 0.40 | 0.43 | 0.45 | 0.68 | 0.82 |

*Source: Own work.*

One can see that in the first sub-period portfolios constructed from the small-cap stocks performed much better than the portfolios from the large-cap stocks. Sharpe ratios are high for three MV portfolios and two MSR portfolios. Looking at the annualized average returns, one can see that the EW portfolio and two MSR portfolios were returning on average more than 10% a year. Annualized returns of the MV portfolios did not lag much and they successfully minimized standard deviation which resulted in them having the highest Sharpe ratios. Turning to the second sub-period, one can see that in terms of annualized average return the MSR portfolio with semi-deviation as a proxy for expected returns and the PCA for covariance matrix performed better than other approaches. It also had the second highest Sharpe ratio. Returns above 10% were registered by the other two MSR portfolios, the MV portfolio with sample covariance matrix and the EW portfolio. In the second sub-period, the MV portfolios again successfully minimized volatility, but they registered lower returns than in the first sub-period (except for the MV portfolio with sample covariance matrix). Sharpe ratio was the highest for the MV portfolio with sample covariance matrix because of the better risk control than the second-best portfolio, which had higher return but also higher risk. In contrast to the large-cap stock portfolios, the betas are now higher in the first sub-period, which can be expected, as systematic risk increases in the market downturns and market factors become more important drivers of stock movements. The only portfolio where this does not hold is the last of the MSR portfolios.

## 4.4 Comparison of the results

Now, that I have examined the performance of different portfolio diversification approaches on the sample of large and small-cap stocks, I also have to take a look at how the results compare among the samples. Therefore, I present the most important portfolio performance measures in Table 10. First, one can see that portfolios constructed from small-cap stocks have higher cumulative returns, average returns and Sharpe ratios. This holds for all portfolios. For the standard deviation, one would expect it to be higher for small-cap stock portfolios as small stocks are more volatile. However, this holds only for the MCW portfolio, the EW portfolio, the FW portfolio and the MSR portfolios with improved estimates. Other portfolios, one MSR portfolio and all the MV portfolios, were able to reduce standard deviation below those of the large-cap portfolios. This means that some of the small-cap stocks must have had really low volatility and the MV portfolios successfully selected those stocks. This lower volatility was also seen from the CAPM analyses, where the MV portfolios from small-cap stocks had betas of 0.43 and 0.44 compared to the portfolios from large-cap stocks which had betas between 0.65 and 0.70.

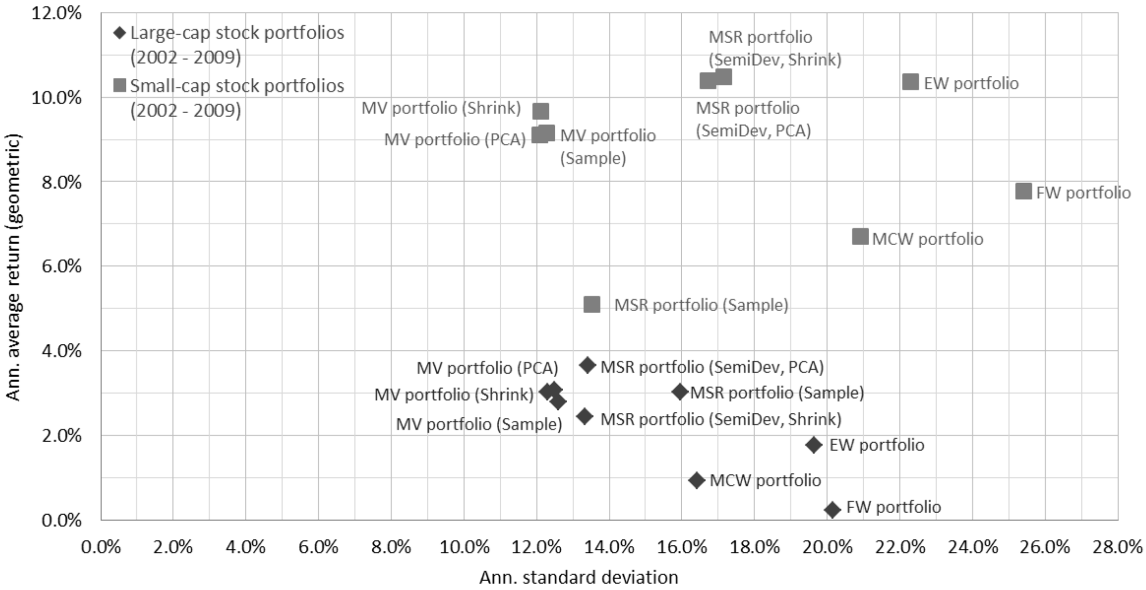
Table 10: Comparison of the European large and small-cap stock portfolios in the period from 2002 till 2018

| | MCW portfolio | EW portfolio | FW portfolio | MV portfolio (Sample) | MV portfolio (PCA) | MV portfolio (Shrink) | MSR portfolio (Sample) | MSR portfolio (SemiDev, PCA) | MSR portfolio (SemiDev, Shrink) |
|---|---|---|---|---|---|---|---|---|---|
| *Large-cap stock portfolios* | | | | | | | | | |
| Cummulative Return | 84.69% | 106.69% | 55.78% | 191.18% | 166.87% | 196.96% | 97.11% | 190.01% | 181.13% |
| Ann. average return (geometric) | 3.67% | 4.36% | 2.64% | 6.49% | 5.94% | 6.61% | 4.07% | 6.46% | 6.27% |
| Ann. standard deviation | 14.07% | 16.56% | 17.65% | 11.51% | 11.23% | 11.31% | 13.27% | 12.23% | 12.25% |
| Sharpe ratio | 0.17 | 0.18 | 0.08 | 0.45 | 0.41 | 0.47 | 0.21 | 0.42 | 0.40 |
| 95% Value at Risk | -6.77% | -7.56% | -8.22% | -5.19% | -5.17% | -5.12% | -6.38% | -5.65% | -5.58% |
| 95% Expected Shortfall | -9.94% | -11.58% | -12.50% | -7.88% | -7.64% | -7.44% | -9.09% | -7.87% | -7.94% |
| Maximum drawdown | -44.92% | -48.98% | -53.76% | -29.72% | -30.94% | -31.06% | -42.08% | -34.01% | -36.16% |
| *Small-cap stock portfolios* | | | | | | | | | |
| Cummulative Return | 273.14% | 418.79% | 245.23% | 385.67% | 306.00% | 300.01% | 254.02% | 545.32% | 464.89% |
| Ann. average return (geometric) | 8.05% | 10.17% | 7.56% | 9.74% | 8.59% | 8.50% | 7.72% | 11.59% | 10.72% |
| Ann. standard deviation | 17.38% | 18.35% | 20.75% | 10.46% | 10.33% | 10.49% | 11.69% | 13.92% | 14.93% |
| Sharpe ratio | 0.39 | 0.48 | 0.30 | 0.80 | 0.70 | 0.68 | 0.55 | 0.73 | 0.63 |
| 95% Value at Risk | -8.22% | -7.24% | -8.34% | -4.70% | -4.61% | -4.53% | -5.70% | -6.32% | -6.65% |
| 95% Expected Shortfall | -14.47% | -11.47% | -13.20% | -7.56% | -6.71% | -6.61% | -9.82% | -10.56% | -10.38% |
| Maximum drawdown | -65.35% | -61.57% | -71.70% | -44.93% | -45.40% | -45.02% | -49.42% | -50.04% | -51.99% |

*Source: Own work.*

Turning to the measures of extreme risk one can see the similar pattern as for the standard deviation in the VaR and ES. These measures are mostly higher for the small-cap stock portfolios, except for the MV portfolios. Although the VaR and ES do not show higher riskiness of the portfolios constructed from small-cap stocks, this is seen from the last measure in the table – maximum drawdown. Looking at this indicator, one can see much higher maximum drawdown of the small-cap stock portfolios, which was around 15% to 20% higher. In this respect, the MV portfolios constructed from the small-cap stocks cannot compare with their counterparts from the large-cap stocks.

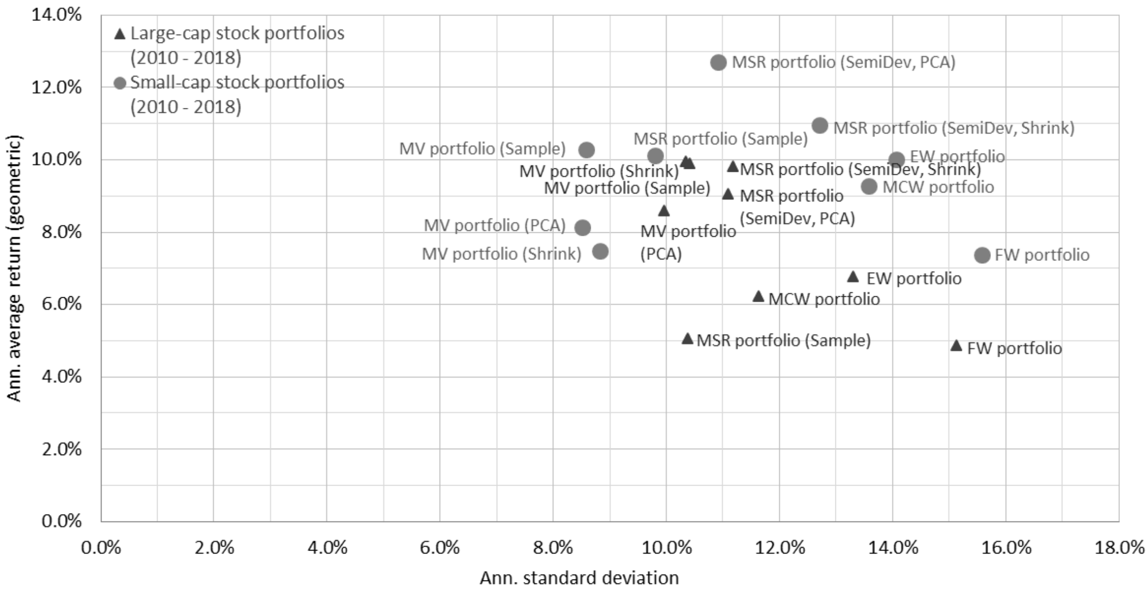
Figure 9: Comparison of the European large and small-cap stock portfolios performance in the period from 2002 till 2009




*Source: Own work.*

In the above figure, one can see the graphical comparison of the portfolio performances in the risk-return space. Results are shown for the large and small-cap stock portfolios in the period from 2002 till 2009 and the differences in returns are enormous. In that period, the best performing portfolio constructed from the large-cap stocks had lower annualized average return than the worst performing portfolio constructed from the small-cap stocks. The standard deviation is on average lower for the large-cap stock portfolios, although the MV portfolios from both samples had very similar volatility.

Figure 10: Comparison of the European large and small-cap stock portfolios performance in the period from 2010 till 2018

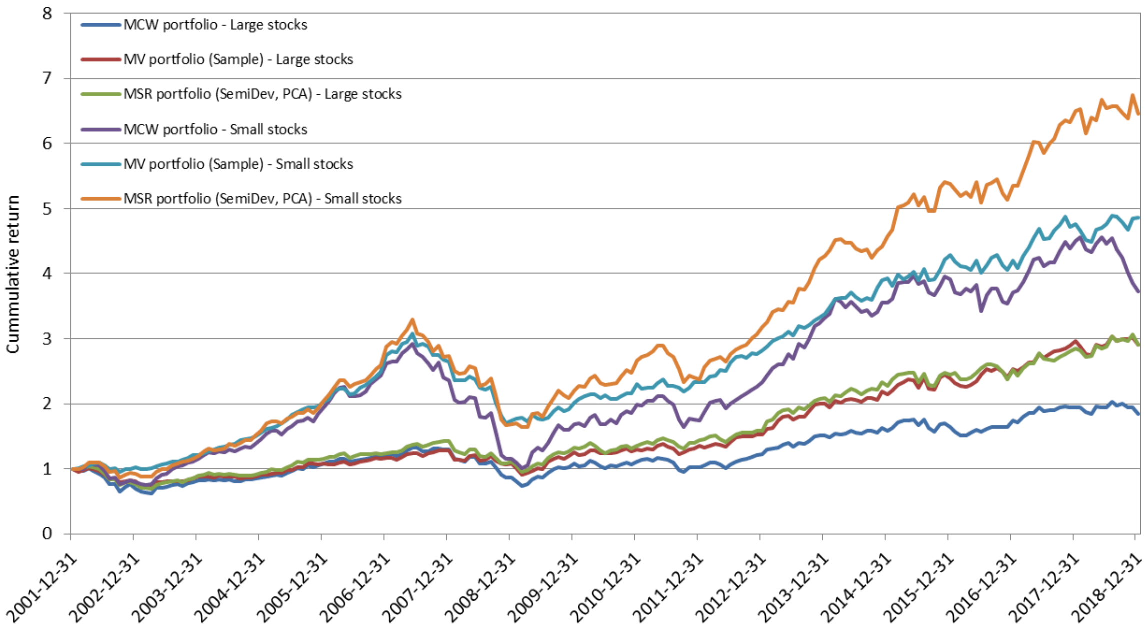



*Source: Own work.*

Figure 10 is a representation of the portfolios' performance in the period from 2010 till 2018. One can see that in the bull markets the differences between portfolios constructed from the small and large-cap stocks are not that big. However, small-cap stock portfolios still offered better risk-adjusted returns. The smaller difference in the large and small-cap stock portfolios performance in the up markets can be explained with the fact that large-cap stocks are usually dragged up in the bull markets as this are the most renowned companies, where the most of investors like to invest. Consequently, this companies exhibit huge growth in stock value and can become overvalued, which is compensated with lower returns in the downmarket environment.

Figure 11 presents the growth of portfolio value (normalised to one as a starting value) for six selected portfolios, three portfolios from each sample, i.e. the MCW portfolio which serves as the benchmark, the MV portfolio with sample estimates and the MSR portfolio with semi-deviation and the PCA. These portfolios were selected as they have the best performance statistics and are representative for the performance of the MV portfolios and the MSR portfolios.

Figure 11: Growth of portfolio value for selected portfolios in the period from 2002 till 2018




*Source: Own work.*

The upper figure shows how the differences in average returns over the years translate into even bigger differences in cumulative returns. This is a great example of how compounding works, e.g. the difference in annualized average return of the MV portfolio and the MSR portfolio constructed from the small-cap stocks is 1.85% points. This small difference in returns compounds into 159.7% points higher cumulative return of the MSR portfolio over the 17 years period. Thus, the selection of the appropriate portfolio is very important as the MCW portfolio of the small-cap stocks had even lower cumulative return that was half lower than the MSR portfolio's (273.14% vs 545.32%).

When taking a quick look at the cumulative returns of the large-cap stock portfolios, one can see that they had much lower cumulative returns. This just confirms that stocks of small companies have more room to grow and over the years this results in higher portfolio wealth. Investors usually turn away from small-cap stocks or allocate only small amounts of portfolio to smaller stocks because they perceive them as risky and miss on the opportunity to grow their portfolio value by considerable amount. Another interesting observation is how closely the MV portfolio and the MSR portfolio of the large-cap stocks moved. This can indicate that stock's semi-deviation better predicts the expected returns of small-cap stocks while it is not that useful for predicting returns of large-cap stocks. This is possible explanation because investors are more concerned about the risk of small-cap stocks and require them to bring higher returns. On the other hand, large-cap stocks are not that volatile and it is thus more difficult to predict the best performing stocks in the next period. However, semi-deviation as the proxy for expected return is still much better than the mean return.

I also have to confirm the research hypotheses that were set in the introduction. Based on the presented results for the small and large-cap stock portfolios and the computed statistical tests presented in Table 2 and Table 6, I confirmed the following hypotheses.

Table 11: Review of the research hypotheses

| | Hypothesis | Result |
|---|---|---|
| 1. | The market-cap weighted portfolio is inefficient as one can construct portfolio with higher out-of-sample Sharpe ratio. | CONFIRMED |
| 2. | The equally-weighted portfolio has higher return and higher Sharpe ratio but also higher risk than the market-cap weighted portfolio. | NOT CONFIRMED |
| 3. | The fundamentally-weighted portfolio has a higher Sharpe ratio than the market-cap weighted portfolio. | NOT CONFIRMED |
| 4. | The minimum variance portfolios with improved estimates of covariance matrix have the lowest standard deviations. | PARTIALLY CONFIRMED |
| 5. | The maximum Sharpe ratio portfolios with improved estimates of the stock return moments have the highest Sharpe ratios. | PARTIALLY CONFIRMED |
| 6. | There are no differences if the approaches are applied to the large or small-cap stocks. | PARTIALLY CONFIRMED |

*Source: Own work.*

The first research hypothesis is confirmed as five portfolios from the large-cap stocks and three portfolios from the small-cap stocks had statistically and economically significant

Sharpe ratios. The MV portfolio with sample covariance matrix and two MSR portfolios with advanced moment estimates had statistically significant result in both stock samples.

The second research hypothesis is not confirmed. The EW portfolio had higher return, higher Sharpe ratio and higher risk in both stock samples, but the differences were not statistically significant.

The third research hypothesis is not confirmed as the FW portfolio did not even have higher Sharpe ratio. Based on the statistical tests for the Sharpe ratio, it is more possible that FW portfolio has lower Sharpe ratio. The p-value was 0.094 which is close to the 0.05 threshold.

The fourth hypothesis is partially confirmed. The MV portfolios had the lowest standard deviations in both samples. The difference seems economically significant for the large-cap stocks but with statistical tests I was not able to confirm that difference. In comparison, the difference was even bigger for small-cap stocks, where it was also statistically significant. At the same time, the MV portfolio with sample covariance matrix minimized volatility as successfully as the MV portfolios with improved estimates of covariance matrix. Based on the mixed results for the small and large-cap stocks, I cannot fully confirm the hypothesis.

The fifth hypothesis is only partially confirmed, although the MSR portfolios with improved portfolio return moments have statistically and economically higher Sharpe ratio than the market-cap weighted portfolio. This difference is present for the small and large-cap stocks. However, I cannot confirm they have the highest Sharpe ratios because in both samples there was at least one MV portfolio with statistically significant difference in Sharpe ratio compared to the MCW portfolio and that Sharpe ratio was higher than those of the MSR portfolios. As a result, I partially confirmed the hypothesis as it improved the Sharpe ratio with respect to the MCW portfolio, but I cannot say that these Sharpe ratios are the highest of all portfolios.

The last hypothesis is only partially confirmed because the portfolio diversification approaches followed the same pattern of improvement of the MCW portfolio, but at the same time, improvements were not statistically significant on both stock samples. For example, on both samples the differences in returns showed similar directions as all portfolios had higher returns than the MCW portfolio, except for the FW portfolio for both samples and MSR portfolio with sample moments for the small-cap stocks. In both samples, the volatility is higher for the EW and the FW portfolios and lower for the optimized portfolio. The Sharpe ratio is also higher for all portfolios expect for the FW portfolio. Thus, in that respect portfolio diversification approaches work the same for small and large-cap stock portfolios. But there is a difference in statistical significance of those results. While differences in returns are not statistically significant for any of the portfolios, the differences in volatilities are significant for the MV portfolios in the small-cap stocks sample but not for the portfolios constructed from the large-cap stocks. Also, the difference in Sharpe ratios is statistically significant for five portfolios in the large-cap stocks sample and three portfolios in the small-cap stocks sample. Therefore, I can only partially confirm the hypothesis.

To sum up, the benefits of using optimized portfolios instead of the simple weighting approaches are huge. The market-cap weighting is inefficient and it should not serve as the proxy for the tangency portfolio, neither should the equal weighting or fundamental weighting with book value of equity. The MV portfolios and the MSR portfolios are thus much better approaches, but the inputs, i.e. the covariance matrix and the expected returns should be estimated with special care. The results show that it is perfectly fine to use the sample covariance matrix for the MV portfolio but I would rather advise to use the PCA as an implicit factor model or shrinkage methods, because if I would have portfolios with more than 50 stocks, the curse of dimensionality and consequently the estimation error would probably step in and result in poorer performance.

When it comes to the maximum Sharpe ratio portfolio, estimation error in expected returns is a huge problem. Therefore, nobody should use the average returns as the inputs for the tangency portfolio. The semi-deviation estimate that I tested and was first proposed by Amenc, Goltz, Martellini and Retkowsky (2010) resulted in the favourable out-of-sample results. However, the out-of-sample Sharpe ratios for the MV portfolios and the MSR portfolios were comparable, which indicates that there is still room for the improvement of the expected return estimates. This improvement can come from the equity research analysts who are able to identify the best performing stocks in the next period. Without better estimates of expected returns minimum variance portfolio and maximum Sharpe ratio portfolio both look like a good proxy for the tangency portfolio – something that was also observed by Amenc, Goltz, Lodh and Martellini (2012).

Two caveats that have to be added to the optimized portfolios are large tracking error and high turnover. The first is very important for portfolio managers, who don not want to deviate too much from their benchmarks. The possible solution to this problem is portfolio optimization, where the maximum allowed tracking error is added as the objective to portfolio optimizer. The problem of high turnover can be solved with optimal portfolio turnover control, where one rebalances portfolio only when there is significant deviation from optimal weights or one sets the maximum allowable portfolio turnover at each rebalancing. These are the question that were not addressed in the thesis as each additional objective reduces the out-of-sample performance to some extent and my goal was to find portfolio with the highest efficiency without additional portfolio constraints.

## CONCLUSION

The market-cap weighting which is the basis for the most equity indexes and benchmarks on which portfolio managers are evaluated has been shown as inefficient weighting approach a long ago. The first, who pointed to this were Haugen and Baker (1991) and Grinold (1992). I first reviewed the portfolio theory, starting from the most basic things such as the portfolio risk and return, covariance between stocks and diversification. I continued with the modern portfolio theory, which was derived by Markowitz (1952) and introduces two remarkable portfolios, the global minimum variance portfolio and the tangency portfolio (maximum Sharpe ratio portfolio). In the absence of the good inputs and sufficient computer power, construction of these portfolios for a large number of stocks was practically impossible at that time. This led to Sharpe (1964) developing the CAPM, where he argued that under certain assumptions the portfolio of all marketable assets weighted by their market value equals the tangency portfolio, and can, therefore, be called the market portfolio. The CAPM has been refuted numerous times as its assumptions are unrealistic and loosening of only one of them results in its sub-optimality. However, in the absence of the better portfolio construction methods investment industry have built numerous stock indexes which are market-cap weighted. The idea of the market-cap weighted indexes as being the best proxy for the tangency portfolio has become firmly rooted in the investment world.

Despite the adoption from the investment industry, which has been introducing more and more cap-weighted indexes, researchers and practitioners started focusing on new approaches to portfolio diversification. I presented the most widely documented approaches in the second chapter. In addition to the market-cap weighting, I presented six other approaches, the two simple portfolio diversification techniques, i.e. the equal weighting and fundamentals weighting, and four advanced approaches, i.e. the minimum variance portfolio, the equally weighted risk contributions portfolio or risk parity, the most diversified portfolio and the maximum Sharpe ratio portfolio.

In the third chapter, I looked at the estimation of portfolio moments, which are the inputs to advanced (optimized) portfolios. The very well-known problem in economics, the "garbage in, garbage out" also applies here. Meaning, no matter how good our models are, if one has poor quality inputs, good results cannot be expected. In this case, the inputs for optimized portfolios are expected returns and variance-covariance matrix. Here, practically all researchers agree that sample covariance matrix and sample expected returns are bad inputs. That is why I reviewed the estimation approaches that should yield better out-of-sample results.

In the last chapter, I conducted the empirical analysis based on the sample of the European large and small-cap stocks. The period of analysis stretches from January 1, 2002 till December 31, 2018. The stock samples were collected from the STOXX Europe 600 index and the period was chosen based on the availability of data on the Bloomberg terminal.

Tested portfolios are not the same as portfolios covered in chapter two of the thesis because the risk parity approach is more useful as the asset class weighting approach and not for equity portfolios. That is also the area where it was developed and the most widely examined. The most diversified portfolio was excluded from the analysis because it produced the good out-of-sample results in only few studies.

In the end, I focused on constructing three simple portfolios (the MCW portfolio, the EW portfolio and the FW portfolio) and six optimized portfolios, the MV portfolio and the MSR portfolio, each estimated with three different combinations of portfolio return moments. I thought it was better to focus on the MV portfolio and the MSR portfolio, which have the strongest theoretical basis, and test them with different inputs instead of focusing on the other weighting approaches which do not have strong theoretical or empirical support. The optimized portfolios with different inputs are:

1. Minimum variance portfolio based on the sample var-cov matrix,
2. Minimum variance portfolio with the PCA used as an estimator of var-cov matrix,
3. Minimum variance portfolio with the shrinkage estimator for var-cov matrix,
4. Maximum Sharpe ratio portfolio based on the sample mean and sample var-cov matrix,
5. Maximum Sharpe ratio portfolio with the semi-deviation used as an estimator for stock expected returns and the PCA as an estimator of var-cov matrix,
6. Maximum Sharpe ratio portfolio with the semi-deviation used as an estimator for stock expected returns and the shrinkage estimator for var-cov matrix.

The main research objective of the empirical analysis was to find the portfolio which improves the efficiency (Sharpe ratio) of the market-cap weighted benchmark. The results based on the samples of small and large-cap stocks showed that the best out-of-sample performance is achieved with the MV portfolios and the MSR portfolios. In both samples the FW portfolio, which was tested only with book value of equity, performed the worst in all aspects and even underperformed the MCW portfolio. The EW portfolio had higher return and higher Sharpe ratio but also higher volatility than the MCW benchmark, but these differences were not statistically significant on any sample. Thus, none of the simple portfolios was able to outperform the benchmark cap-weighted portfolio by statistically significant margin.

The results for the optimized portfolios on the large-cap sample showed statistically significant improvement in Sharpe ratios for three MV portfolios and two MSR portfolios with improved estimates. The differences in returns and volatilities were not statistically significant. The small-cap sample showed slightly different results as the statistically significant difference in Sharpe ratios was found only for the MV portfolio with sample covariance matrix and two MSR portfolios with improved input estimates. Additionally, statistically significant difference in volatilities was obtained for all MV portfolios. The differences in returns were not statistically significant.

The conclusions I have made are, that MSR portfolio should only be constructed with improved estimates of portfolio return moments as sample mean is a bad estimate of expected returns. The MV portfolio did not show any problems when sample covariance matrix was used, however, I would still advise to rather use methods such as the PCA or shrinkage estimators. That is because the problem of dimensionality in covariance matrix estimation process comes into play when one has larger stock portfolios. My portfolio of 50 stocks is much smaller than portfolios with 500 or more stocks, where the number of required estimates grows exponentially.

Additionally, the optimized portfolios exhibit high tracking error and turnover which can be problematic for portfolio managers. The solution to this problem is optimization with additional objective functions, such as the maximum allowed tracking error and optimal portfolio turnover control. The later can be added as the maximum allowed turnover at each rebalancing or one can rebalance only when the weights deviate significantly from the optimal weights. Diversification of optimal portfolios measured as the effective number of stocks in the portfolio is also problematic as optimizers pick only a handful of stocks. This can be improved with additional minimum and maximum weight constraints.

The out-of-sample Sharpe ratios of the MV portfolios and the MSR portfolios also look very similar, which was noted also by Amenc, Goltz, Lodh and Martellini (2012). Therefore, both portfolios look like a good proxy for the tangency portfolio. This indicates that active portfolio managers and stock research analysts can still beat those portfolios if they can come up with better estimates of expected returns. But in the absence of the better estimates, investors can divide their money between the MV portfolio and the MSR portfolio as they have different characteristics and exhibit different performance in bear and bull markets.

When it comes to the differences between samples it is evident that small-cap stocks had much higher returns and over the years this resulted in almost twice the cumulative returns. At the same time, the MV portfolios from small-cap stocks even had lower volatility than the MV portfolios form large-cap stocks. Also, the extreme risk measures did not show higher riskiness of small-cap stock portfolios until I looked at the maximum drawdown which was much higher for all small-cap stock portfolios.

Based on the obtained results, which are similar to other comparable studies, efficient portfolio diversification can only be achieved with optimized portfolios, namely, the minimum variance portfolio and the maximum Sharpe ratio portfolio, where it is important to use improved methods for the estimation of expected returns and variance-covariance matrix. Limiting the investment universe only to large-cap stocks can result in lower cumulative returns over the longer investment periods as small-cap stocks compensate their higher riskiness with much higher returns.

## REFERENCE LIST


1. Amenc, N., Goltz, F., Lodh, A. & Martellini L. (2012). Diversifying the Diversifiers and Tracking the Tracking Error: Outperforming Cap-Weighted Indices with Limited Risk of Underperformance. *The Journal of Portfolio Management, 38*(3), 72–88.
2. Amenc, N., Goltz, F., Martellini, L. & Retkowsky, P. (2010). Efficient Indexation: An Alternative to Cap-Weighted Indices. *Journal of investment management, 9*(4), 1–23.
3. Ang, A., Hodrick, R. J., Xing, Y. & Zhang, X. (2006). The Cross-Section of Volatility and Expected Returns. *Journal of Finance, 61*(1), 259–299.
4. Ardia, D. & Boudt, K. (2018). 'PeerPerformance': Luck-corrected peer performance analysis in 'R'. *R package version 2.2.1*. Retrieved April 1, 2019, from https://CRAN.R-project.org/package=PeerPerformance
5. Ardia, D., Mullen, K. M., Peterson, B. G. & Ulrich, J. (2016). 'DEoptim': Differential Evolution in 'R'. *R package version 2.2-4*. Retrieved March 2, 2019, from https://CRAN.R-project.org/package=DEoptim
6. Arnott, R., Hsu, J. & Moore, P. (2005). Fundamental Indexation. *Financial Analysts Journal, 61*(2), 83–99.
7. Arnott, R., Kalesnik, V., Moghtader, P. & Scholl, C. (2010). Beyond Cap Weight. *Journal of Indexes, 13*(1), 16–29.
8. Bartholdy, J. & Peare, P. (2005). Estimation of expected return: CAPM vs. Fama and French. *International Review of Financial Analysis, 14*(4), 407–427.
9. Bennett, R. (2015). Advanced Moment Estimation & Optimization with PortfolioAnalytics. Retrieved March 26, 2019, from http://rossb34.github.io/PortfolioAnalyticsPresentation2015/#1
10. Bielstein, P. & Hanauer, M. X. (2018). Mean-variance optimization using forward-looking return estimates. *Review of Quantitative Finance and Accounting,*1–26.
11. Black, F. & Litterman, R. (1992). Global Portfolio Optimization. *Financial Analysts Journal, 48*(5), 28–43.
12. Blitz, D. & Swinkels, L. (2008). Fundamental indexation: An active value strategy in disguise. *Journal of Asset Management, 9*(4), 264–269.
13. Bloomberg L.P. (2019). *STOXX Europe 600 constituent list as of December 31, 2001 with stock's market capitalizations, book value of equity, monthly returns and German 1-year government bond returns, from 12/31/1997 to 12/31/2018*. Retrieved March 11, 2019 from Bloomberg terminal.
14. Bodie, Z., Kane, A. & Marcus, A. J. (2011). *Investments* (9th ed.). New York: McGraw-Hill.
15. Boyer, B., Mitton, T. & Vorkink, K. (2010). Expected Idiosyncratic Skewness. *The Review of Financial Studies, 23*(1), 169–202.
16. Breeden, D. T. (1979). An intertemporal asset pricing model with stochastic consumption and investment opportunities. *Journal of Financial Economics, 7*(3), 265–296.

17. Brennan, M. J. (1970). Taxes, market valuation and corporate financial policy. *National Tax Journal, 23*(4), 417–427.
18. Britten-Jones, M. (1999). The Sampling Error in Estimates of Mean-Variance Efficient Portfolio Weights. *Journal of Finance, 54*(2), 655–671.
19. Cagna, E. & Casuccio, G. (2014). Equally-weighted Risk Contribution Portfolios: an empirical study using expected shortfall. *Center for research on pensions and welfare policies*, working paper 142/14.
20. Carhart, M. M. (1997). On Persistence in Mutual Fund Performance. *The Journal of Finance, 52*(1), 57–82.
21. Chan, L. K. C., Karceski, J. & Lakonishok. (1999). On portfolio optimization: Forecasting covariances and choosing the risk model. *Review of Financial Studies, 12*(5), 937–974.
22. Chen, D. H., Chen, C. D. & Chen, J. (2009). Downside risk measures and equity returns in the NYSE. *Applied Economics, 41*(8), 1055–1070.
23. Chopra, V. K. & Ziemba, W. T. (1993). The Effect of Errors in Means, Variances, and Covariances on Optimal Portfolio Choice. *Journal of Portfolio Management, 19*(2), 6–11.
24. Choueifaty, Y. & Coignard, Y. (2008). Toward Maximum Diversification. *The Journal of Portfolio Management, 35*(1), 40–51.
25. Choueifaty, Y., Froidure, T. & Reynier, J. (2013). Properties of the Most Diversified Portfolio. *Journal of Investment Strategies, 2*(2), 49–70.
26. Chow, T. M., Hsu, J. C., Kuo, L.L. & Li, F. (2014). A Study of Low-Volatility Portfolio Construction Methods. *The Journal of Portfolio Management, 40*(4), 89–105.
27. Clarke, R., De Silva, H. & Thorley, S. (2006). Minimum-Variance Portfolios in the U.S. Equity Market. *The Journal of Portfolio Management, 33*(1), 10–24.
28. Clarke, R., De Silva, H. & Thorley, S. (2011). Minimum-Variance Portfolio Composition. *The Journal of Portfolio Management, 37*(2), 31–45.
29. Clarke, R., De Silva, H. & Thorley, S. (2013). Risk Parity, Maximum Diversification, and Minimum Variance: An Analytic Perspective. *The Journal of Portfolio Management, 39*(3), 39–53.
30. Cuthbertson, K. & Nitzsche, D. (2004). *Quantitative financial economics: Stocks, bonds and foreign exchange* (2nd ed.). Chichester, England: Wiley.
31. DeMiguel, V., Garlappi, L. & Uppal, R. (2009). Optimal Versus Naive Diversification: How Inefficient is the 1/N Portfolio Strategy?. *The Review of Financial Studies, 22*(5), 1915–1953.
32. Dittmar, R. (2002). Nonlinear Pricing Kernels, Kurtosis Preference, and Evidence from the Cross Section of Equity Returns. *Journal of Finance, 57*(1), 369–403.
33. Dragulescu, A. A. & Arendt, C. (2018). xlsx: Read, Write, Format Excel 2007 and Excel 97/2000/XP/2003 Files. *R package version 0.6.1*. Retrieved March 11, 2019 from https://CRAN.R-project.org/package=xlsx

34. Evans, J. L. & Archer, S. H. (1968). Diversification and the reduction of dispersion: an empirical analysis. *The Journal of Finance, 23*(5), 761–767.
35. Fama, E. F. & French, K. R. (1992). The Cross-Section of Expected Stock Returns. *Journal of Finance, 47*(2), 427–465.
36. Fama, E. F. & French, K. R. (1993). Common risk factors in the returns on stocks and bonds. *Journal of Financial Economics, 33*(1), 3–56.
37. Fama, E. F. & French, K. R. (1997). Industry costs of equity. *Journal of Financial Economics*, *43*(2), 153–193.
38. Fama, E. F. & French, K. R. (2015). A five-factor asset pricing model. *Journal of Financial Economics, 116*(1), 1–22.
39. Foye, J., Mramor, D. & Pahor, M. (2013). A Respecified Fama French Three-Factor Model for the New European Union Member States. *Journal of International Financial Management & Accounting, 24*(1), 3–25.
40. Fujiwara, Y., Souma, W., Murasato, H. & Yoon, H. (2006). Application of PCA and Random Matrix Theory to Passive Fund Management. *Practical Fruits of Econophysics*, 226–230. Tokyo: Springer.
41. Goltz, F. & Le Sourd, V. (2011). Does Finance Theory Make the Case for Capitalisation-Weighted Indexing?. *The Journal of Index Investing, 2*(2), 59–75.
42. Gosier, K., Madhavan, A., Serbin, V. & Yang, J. (2005). Toward Better Risk Forecasts. *Journal of Portfolio Management, 31*(3), 82–91.
43. Griffin, J. M. (2002). Are the Fama and French Factors Global or Country Specific?. *The Review of Financial Studies, 15*(3), 783–803.
44. Grinold, R. (1992). Are Benchmark Portfolios Efficient?. *The Journal of Portfolio Management, 19*(1), 34–40.
45. Harvey, C. R., Liechty, J. C. & Liechty, M. W. (2008). Bayes vs. Resampling: A Rematch. *Journal of Investment Management, 6*(1), 1–17.
46. Haugen, R. & Baker, N. (1991). The efficient market inefficiency of capitalization-weighted stock portfolios. *The Journal of Portfolio Management, 17*(3), 35–40.
47. Hemminki, J. & Puttonen. (2008). Fundamental indexation in Europe. *Journal of Asset Management, 8*(6), 401–405.
48. Heaton, J. & Lucas, D. (2000). Portfolio Choice and Asset Prices: The Importance of Entrepreneurial Risk. *The Journal of Finance, 55*(3), 1163–1198.
49. Hornik, K., Meyer, D., Schwendinger, F. & Theussl, S. (2019). ROI: R Optimization Infrastructure. *R package version 0.3-2*. Retrieved March 20, 2019 from https://CRAN.R-project.org/package=ROI
50. Hsu, J. C. (2006). Cap-Weighted Portfolios are Sub-Optimal Portfolios. *Journal of Investment Management, 4*(3), 1–10.
51. Hsu, J. C. & Campollo, C. (2006, January/February). New Frontiers in Index Investing. An Examination of Fundamental Indexation. *Journal of Indexes, 58*, 32–37.
52. Investopedia. (n.d.). Expected Return, Variance And Standard Deviation Of A Portfolio. Retrieved December 5, 2018, from

https://www.investopedia.com/walkthrough/corporate-finance/4/return-risk/expected-return.aspx
53. Jagannathan, R. & Ma, T. (2003). Risk Reduction in Large Portfolios: Why Imposing the Wrong Constraints Helps. *Journal of Finance, 58*(4), 1651–1683.
54. Jensen, M. C., Black, F. & Scholes, M. S. (1972). The Capital Asset Pricing Model: Some Empirical Tests. *Studies in the Theory of Capital Markets*. New York: Praeger.
55. Jorion, P. (1985). International Portfolio Diversification with Estimation Risk. *Journal of Business, 58*(3), 259–278.
56. Jorion, P. (1986). Bayes-Stein Estimation for Portfolio Analysis. *Journal of Financial and Quantitative Analysis, 21*(3), 279–292.
57. Kaplan, P. D. (2008). Why Fundamental Indexation Might-or Might Not-Work. *Financial Analysts Journal, 64*(1), 32–39.
58. Kassam, A., Gupta, A., Jain, S., Kouzmenko, R. & Briand, R. (2013). Introducing MSCI IndexMetrics. *MSCI Inc*. Retrieved March 28, 2019, from https://www.msci.com/documents/10199/402635a5-fd5d-498e-985a-1bec8ff8d8b1
59. Kenton, W. (2019a). Sortino Ratio Definition. *Investopedia*. Retrieved March 28, 2019, from https://www.investopedia.com/terms/s/sortinoratio.asp
60. Kenton, W. (2019b). Treynor Ratio. *Investopedia*. Retrieved March 28, 2019, from https://www.investopedia.com/terms/t/treynorratio.asp
61. Kraus, A. & Litzenberger, R. (1976). Skewness Preference and the Valuation of Risk Assets. *Journal of Finance, 31*(4), 1085–1100.
62. Lintner, J. (1965). The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets. *The Review of Economics and Statistics, 47*(1), 13–37.
63. Ledoit, O. & Wolf, M. (2003). Improved estimation of the covariance matrix of stock returns with an application to portfolio selection. *Journal of Empirical Finance, 10*(5), 603–621.
64. Ledoit, O. & Wolf, M. (2004). Honey, I Shrunk the Sample Covariance Matrix. *Journal of Portfolio Management, 30*(4), 110–119.
65. Maillard, S., Roncalli, T. & Teiletche, J. (2010). The Properties of Equally Weighted Risk Contribution Portfolios. *The Journal of Portfolio Management, 36*(4), 60–70.
66. Malkiel, B. (1995). Returns from Investing in Equity Mutual Funds 1971 to 1991. *Journal of Finance, 50*(2), 549–572.
67. Malkiel, B. & Xu, Y. (1997). Risk and Return Revisited. *Journal of Portfolio Management, 23*(3), 9–14.
68. Malkiel, B. & Xu, Y. (2006). Idiosyncratic Risk and Security Returns. *Working paper*. Richardson: University of Texas at Dallas.
69. Malladi, R. & Fabozzi, F. J. (2017). Equal-weighted strategy: Why it outperforms value-weighted strategies? Theory and evidence. *Journal of Asset Management, 18*(3), 188–208.
70. Markowitz, H. (1952). Portfolio Selection. *The Journal of Finance, 7*(1), 77–91.

71. Markowitz, H. (1956). The optimization of a quadratic function subject to linear constraints. *Naval Research Logistics Quarterly, 3*(1–2), 111–133.
72. Markowitz, H. (1991). Foundations of Portfolio Theory. *The Journal of Finance, 46*(2), 469–477.
73. Markowitz, H. & Usmen, N. (2003). Resampled Frontiers Versus Diffuse Bayes: An Experiment. *Journal of Investment Management, 1*(4), 9–25.
74. Martellini, L. (2008). Towards the Design of Better Equity Benchmarks: Rehabilitating the tangency portfolio from modern portfolio theory. *The Journal of Portfolio Management*, *34*(4), 34–41.
75. Martellini, L. & Ziemann, V. (2010). Improved Estimates of Higher-Order Comoments and Implications for Portfolio Selection. *The Review of Financial Studies*, *23*(4), 1467–1502.
76. Mayers, D. (1973). Nonmarketable Assets and the Determination of Capital Asset Prices in the Absence of a Riskless Asset. *The Journal of Business*, *46*(2), 258–267.
77. Merton, R. C. (1973). An Intertemporal Capital Assets Pricing Model. *Econometrica, 41*(5), 867–887.
78. Merton, R. C. (1980). On estimating the expected return on the market: An exploratory investigation. *Journal of Financial Economics*, *8*(4), 323–361.
79. Michaud, R. (1989). The Markowitz Optimization Enigma: Is 'Optimized' Optimal?. *Financial Analysts Journal, 45*(1), 31–42.
80. Michaud, R. & Michaud, R. (1998). *Efficient Asset Management: A Practical Guide to Stock Portfolio Optimization and Asset Allocation* (2nd ed.). Oxford: Oxford University Press.
81. Michaud, R. & Michaud, R. (2008). Estimation Error and Portfolio Optimization: A Resampling Solution. *Journal of Investment Management*, *6*(1), 8–28.
82. Mitton, T. & Vorkink, K. (2007). Equilibrium Underdiversification and the Preference for Skewness. *Review of Financial Studies*, *20*(4), 1255–1288.
83. Mossin, J. (1966). Equilibrium in a Capital Asset Market. *Econometrica*, *34*(4), 768–783.
84. Pastor, L., Sinha, M. & Swaminathan, B. (2008). Estimating the Intertemporal Risk-Return Tradeoff Using the Implied Cost of Capital. *Journal of Finance*, *63*(6), 2859–2897.
85. Perold, A. F. (2007). Fundamentally Flawed Indexing. *Financial Analysts Journal, 63*(6), 31–37.
86. Peterson, B. G. & Carl, P. (2018a). PerformanceAnalytics: Econometric Tools for Performance and Risk Analysis. *R package version 1.5.2*. Retrieved February 27, 2019 from https://CRAN.R-project.org/package=PerformanceAnalytics
87. Peterson, B. G. & Carl, P. (2018b). PortfolioAnalytics: Portfolio Analysis, Including Numerical Methods for Optimization of Portfolios. *R package version 1.1.0*. Retrieved March 20, 2018 from https://CRAN.R-project.org/package=PortfolioAnalytics

88. Plyakha, Y., Uppal, R. & Vilkov, G. (2012). Why Does an Equal-Weighted Portfolio Outperform Value- and Price-Weighted Portfolios. *EDHEC-Risk Institute*.
89. Plyakha, Y., Uppal, R. & Vilkov, G. (2014). Equal or Value Weighting? Implications for Asset-Pricing Tests. *EDHEC-Risk Institute*.
90. Plyakha, Y., Uppal, R. & Vilkov, G. (2015).Why Do Equal-Weighted Portfolios Outperform Value-Weighted Portfolios?. *SSRN Electronic Journal*.
91. Qian, E. (2005). Risk Parity Portfolios: Efficient Portfolios Through True Diversification. *PanAgora Asset Management*. Retrieved January 17, 2019, from https://www.panagora.com/assets/PanAgora-Risk-Parity-Portfolios-Efficient-Portfolios-Through-True-Diversification.pdf
92. Qian, E. (2006). On the Financial Interpretation of Risk Contribution: Risk Budgets Do Add Up. *Journal of Investment Management, 4*(4), 41–51.
93. Roll, R. (1977). A critique of the asset pricing theory's tests Part I: On past and potential testability of the theory. *Journal of Financial Economics, 4*(2), 129–176.
94. RStudio Team (2016). RStudio: Integrated Development for R (Version 1.1.463). *RStudio, Inc.* Boston, MA. Retrieved October 18, 2018, from http://www.rstudio.com/
95. Ryan, J. A. Ulrich, J. M. (2018). xts: eXtensible Time Series. *R package version 0.11-2*. Retrieved February 22, 2019 from https://CRAN.R-project.org/package=xts
96. Sharpe, W. (1963). A Simplified Model for Portfolio Analysis. *Management Science, 9*(2), 277–293.
97. Sharpe, W. (1964). Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk. *The Journal of Finance, 19*(3), 425–442.
98. Shefrin, H. & Statman, M. (2000). Behavioral Portfolio Theory. *The Journal of Quantitative Analysis, 35*(2), 127–151.
99. Statman, M. (1987). How Many Stocks Make a Diversified Portfolio?. *Journal of Financial and Quantitative Analysis, 22*(3), 353–363.
100. Stefanovits, D. (2010). *Equal Contributions to Risk and Portfolio Construction* (master thesis). ETH Zurich.
101. Stein, C. (1955). Inadmissibility of the usual estimator for the mean of a multivariate normal distribution. *Proceedings of the third Berkeley Symposium on Mathematical Statistics and Probability, 1,* 197–206. Berkeley: University of California Press.
102. Stoxx. (n.d.). STOXX® Europe 600. Retrieved March 26, 2019, from https://www.stoxx.com/index-details?symbol=SXXGR
103. Sushko, V. & Turner, G. (2018). The implications of passive investing for securities markets. *BIS Quarterly Review,* March 2018, 113–131.
104. Tang, Gordon, Y.N. (2004). How efficient is naïve portfolio diversification? an educational note. *Omega, 32*(2), 155–160.
105. Tobin, J. (1958). Liquidity Preference as Behavior Towards Risk. *The Review of Economic Studies, 25*(2), 65–86.
106. Treynor, J. (2005). Why Market-Valuation-Indifferent Indexing Works. *Financial Analysts Journal, 61*(5), 65–69.

107. Zeileis, A. & Grothendieck, G. (2005). zoo: S3 Infrastructure for Regular and Irregular Time Series. *Journal of Statistical Software*, 14(6), 1–27.

# APPENDICES

## Appendix 1: Povzetek (Summary in Slovene language)

V magistrski nalogi se ukvarjam z učinkovito razpršitvijo portfelja evropskih delnic z visoko in nizko tržno kapitalizacijo, saj je bilo dokazano, da je trenutno prevladujoč način izgradnje portfelja, ki temelji na ponderiranju delnic glede na njihovo tržno vrednost, neučinkovit. To sta kot prva dokazala Haugen in Baker (1991) ter Grinold (1992).

S tem namenom prvo pregledam teorijo portfelja, začenši s pregledom osnovnih značilnosti portfelja in učinki preproste razpršitve portfelja. Nato nadaljujem z moderno teorijo portfelja, ki jo je postavil Markowitz (1952), ter v njej definiral dva portfelja – portfelj z najnižjo varianco in portfelj z najvišjim Sharpovim kazalnikom. Sharpov kazalnik meri razmerje med presežnim donosom portfelja nad netvegano stopnjo donosa deljeno s standardnim odklonom portfelja (enačba 7). Omenjeni portfelj dobimo tako, da pri vsakem standardnem odklonu poiščemo portfelj z najvišjim donosom, pri čemer je najboljši tisti, ki doseže najvišji Sharpov kazalnik. Ta portfelj je v teoriji najboljša (najbolj učinkovita) izbira za vsakega vlagatelja, vendar pa obstajajo težave pri njegovi implementaciji v praksi, predvsem je potrebno natančno oceniti prihodnje donose vseh delnic ter variančno-kovariančno matriko. Sharpe (1964) je v modelu vrednotenja dolgoročnih naložb prikazal, kako pod določenimi predpostavkami ta portfelj postane tržni portfelj, kjer so vse delnice ponderirane glede na njihovo tržno vrednost. Ta način izgradnje portfeljev, ki je veliko enostavnejši, prevladuje še danes, v njih pa se vlaga vedno več denarja, kljub vprašljivi učinkovitosti.

V drugem poglavju magistrske naloge pregledam še ostale pristope pri ponderiranju delnic v portfelju, ki so bili predlagani s strani akademikov in upravljavcev premoženja, kot možne alternative tržno ponderiranim portfeljem. Ti portfelji so: portfelj z enakimi utežmi, portfelji ponderirani z vrednostmi knjigovodskih postavk podjetij, portfelj z najnižjo varianco, portfelj ponderiran z enakomernimi prispevki k tveganju, najbolj razpršen portfelj in portfelj z najvišjim Sharpovim kazalnikom.

Portfelji, ki temeljijo na optimizaciji, tj. portfelj z najnižjo varianco in portfelj z najvišjim Sharpovim kazalnikom, zahtevajo ocene bodočih donosov delnic in oceno variančno-kovariančne matrike. Tu nastane problem, kako čim bolj natančno oceniti te podatke, zato v tretjem poglavju pregledam različne metode za njihovo čim bolj natančno oceno, saj povprečne vrednosti iz preteklosti preslabo napovedujejo prihodnost, kar se odrazi v slabi učinkovitosti portfelja.

V zadnjem delu magistrske naloge analiziram 9 različnih portfeljev, ki so: tržni portfelj, portfelj z enakimi utežmi, portfelj ponderiran s knjigovodsko vrednostjo kapitala podjetij ter portfelja z najnižjo varianco in z najvišjim Sharpovim kazalnikom, ki sta ocenjena vsak po trikrat, pri čemer so uporabljene različne metode za oceno prihodnjih donosov delnic in variančno-kovariančne matrike. Analiza je narejena na dveh vzorcih evropskih delnic z visoko in nizko tržno kapitalizacijo, vsak s 50 delnicami, ki so bile izbrane iz indeksa STOXX Europe 600, med 1. januarjem 2002 in 31. decembrom 2018. Testiranje je bilo izvedeno za nazaj, uporabljeni so bili podatki o mesečnih donosih, optimizirana portfelja sta imela omejitev glede najnižje in najvišje uteži posamezne delnice, in sicer 0 % in 20 %, portfelj pa je moral biti investiran v celoti.

Dobljeni rezultati so podobni ugotovitvam preteklih študij, ki so bile izvede pretežno za ameriški delniški trg, in kažejo, da so portfelji z najvišjim Sharpovim kazalnikom, ki uporabljajo izboljšane metode za ocene prihodnjih donosov in variančno-kovariančne matrike, veliko boljši od trenutno prevladujočih tržnih portfeljev. Ugotovljene razlike v Sharpovih kazalnikih so statistično značilne na obeh vzorcih. Vsi portfelji z najnižjo varianco so dosegli višji Sharpov kazalnik na vzorcu velikih delnic, medtem ko je bil na vzorcu malih delnic kazalnik statistično značilen le za portfelj z vzorčno variančno-kovariančno matriko, kar nam pove, da pri portfelju s 50-imi delnicami ne obstaja problem prevelike dimenzionalnosti kovariančne matrike in ni potrebno uporabiti izboljšanih metod. Na vzorcu delnic z nizko kapitalizacijo je bila statistično značilna tudi razlika v standardnem odklonu portfeljev z najnižjo varianco, kjer so ti portfelji uspeli znatno zmanjšati volatilnost napram tržnemu portfelju. Vsi ostali portfelji niso izboljšali rezultata tržnega portfelja z razliko, ki bi bila statistično značilna. Portfelj ponderiran s knjigovodsko vrednostjo kapitala podjetij se je odrezal celo slabše v vseh pogledih. Rezultati so predstavljeni v tabelah od 2 do 10.

Razlike med vzorcema kažejo predvsem na višjo donosnost portfeljev sestavljenih iz delnic z nizko tržno kapitalizacijo, vendar pa so ti portfelji podvrženi večjemu ekstremnemu tveganja, saj so delnice z nizko kapitalizacijo bolj tvegane. Kljub temu pa je razlika v kumulativnem donosu lahko dvakrat večja, če primerjamo enak portfelj sestavljen iz delnic z nizko in visoko tržno kapitalizacijo. Delnice majhnih podjetij je torej smiselno vključiti v vsak portfelj. Razlike so razvidne iz slik 9, 10 in 11.

Na podlagi rezultatov je učinkovito razpršitev portfelja možno doseči le z optimiziranimi portfelji, in sicer s portfeljem z najnižjo varianco in portfeljem z najvišjim Sharpovim kazalnikom, kjer je pomembno uporabiti izboljšane metode za ocenjevanje pričakovanih donosov in variančno-kovariančne matrike. Omejevanje portfelja le na delnice z visoko tržno kapitalizacijo vpliva na nižje kumulativne donose skozi daljše časovno obdobje, saj delnice z nizko tržno kapitalizacijo nadomestijo višjo tveganost z večjimi donosi.

## Appendix 2: Sample of the European large-cap stocks

| Bloomberg ticker | Company name | Market cap (EUR million) |
|---|---|---|
| VOD LN Equity | Vodafone Group PLC | 199,340 |
| BP/ LN Equity | BP PLC | 197,097 |
| GSK LN Equity | GlaxoSmithKline PLC | 174,994 |
| NOKIA FH Equity | Nokia OYJ | 137,087 |
| HSBA LN Equity | HSBC Holdings PLC | 122,859 |
| NOVN SW Equity | Novartis AG | 116,924 |
| FP FP Equity | TOTAL SA | 113,098 |
| NESN SW Equity | Nestle SA | 92,653 |
| AZN LN Equity | AstraZeneca PLC | 88,460 |
| DTE GY Equity | Deutsche Telekom AG | 81,017 |
| RBS LN Equity | Royal Bank of Scotland Group PLC | 77,820 |
| ROG SW Equity | Roche Holding AG | 69,201 |
| LLOY LN Equity | Lloyds Banking Group PLC | 68,775 |
| VIV FP Equity | Vivendi SA | 66,746 |
| SIE GY Equity | Siemens AG | 66,040 |
| UNA NA Equity | Unilever NV | 64,933 |
| ULVR LN Equity | Unilever PLC | 64,321 |
| ALV GY Equity | Allianz SE | 64,157 |
| BARC LN Equity | Barclays PLC | 61,573 |
| SAN FP Equity | Sanofi | 61,340 |
| CSGN SW Equity | Credit Suisse Group AG | 56,835 |
| INGA NA Equity | ING Groep NV | 56,435 |
| ENI IM Equity | Eni SpA | 56,222 |
| MUV2 GY Equity | Munich Re | 53,959 |
| OR FP Equity | L'Oreal SA | 53,310 |
| ERICB SS Equity | Telefonaktiebolaget LM Ericsson | 49,540 |
| DBK GY Equity | Deutsche Bank AG | 49,346 |
| LHN SW Equity | LafargeHolcim Ltd | 48,647 |
| SAP GY Equity | SAP SE | 46,043 |
| BNP FP Equity | BNP Paribas SA | 44,520 |
| DGE LN Equity | Diageo PLC | 43,485 |
| AGN NA Equity | Aegon NV | 43,236 |
| PHIA NA Equity | Koninklijke Philips NV | 42,505 |
| CA FP Equity | Carrefour SA | 41,531 |
| CS FP Equity | AXA SA | 40,288 |
| G IM Equity | Assicurazioni Generali SpA | 39,763 |
| EOAN GY Equity | E.ON SE | 38,524 |
| BT/A LN Equity | BT Group PLC | 35,829 |
| RIO LN Equity | Rio Tinto PLC | 33,513 |
| STM FP Equity | STMicroelectronics NV | 32,413 |
| AV/ LN Equity | Aviva PLC | 31,041 |
| AD NA Equity | Koninklijke Ahold Delhaize NV | 30,046 |
| TSCO LN Equity | Tesco PLC | 28,194 |
| GLE FP Equity | Societe Generale SA | 27,068 |
| BAYN GY Equity | Bayer AG | 26,146 |
| PRU LN Equity | Prudential PLC | 25,743 |
| BMW GY Equity | Bayerische Motoren Werke AG | 25,684 |
| BAS GY Equity | BASF SE | 24,737 |
| RWE GY Equity | RWE AG | 23,328 |
| SKY LN Equity | Sky Ltd | 23,322 |

*Source: Bloomberg L.P. (2019)*

## Appendix 3: Sample of the European small-cap stocks

| Bloomberg ticker | Company name | Market cap (EUR million) |
|---|---|---|
| RCH LN Equity | Reach PLC | 1,934 |
| TOM NO Equity | Tomra Systems ASA | 1,925 |
| CON GY Equity | Continental AG | 1,867 |
| MN IM Equity | Arnoldo Mondadori Editore SpA | 1,841 |
| COLR BB Equity | Colruyt SA | 1,821 |
| STB NO Equity | Storebrand ASA | 1,812 |
| G1A GY Equity | GEA Group AG | 1,805 |
| CBG LN Equity | Close Brothers Group PLC | 1,796 |
| COLOB DC Equity | Coloplast A/S | 1,775 |
| COB LN Equity | Cobham PLC | 1,762 |
| GFC FP Equity | Gecina SA | 1,760 |
| PSN LN Equity | Persimmon PLC | 1,735 |
| RAND NA Equity | Randstad NV | 1,727 |
| TPEIR GA Equity | Piraeus Bank SA | 1,668 |
| SOON SW Equity | Sonova Holding AG | 1,660 |
| SBMO NA Equity | SBM Offshore NV | 1,647 |
| TITK GA Equity | Titan Cement Co SA | 1,638 |
| BDEV LN Equity | Barratt Developments PLC | 1,633 |
| SON PL Equity | Sonae SGPS SA | 1,620 |
| SGC LN Equity | Stagecoach Group PLC | 1,594 |
| OERL SW Equity | OC Oerlikon Corp AG | 1,592 |
| IMI LN Equity | IMI PLC | 1,544 |
| TPK LN Equity | Travis Perkins PLC | 1,541 |
| JDW LN Equity | J D Wetherspoon PLC | 1,502 |
| GN DC Equity | GN Store Nord A/S | 1,478 |
| BKG LN Equity | Berkeley Group Holdings PLC | 1,468 |
| BPI PL Equity | Banco BPI SA | 1,459 |
| BPSO IM Equity | Banca Popolare di Sondrio SCPA | 1,444 |
| DLAR LN Equity | De La Rue PLC | 1,430 |
| SGSN SW Equity | SGS SA | 1,408 |
| HWDN LN Equity | Howden Joinery Group PLC | 1,360 |
| PNN LN Equity | Pennon Group PLC | 1,352 |
| BTG LN Equity | BTG PLC | 1,286 |
| NEX LN Equity | National Express Group PLC | 1,214 |
| INM ID Equity | Independent News & Media PLC | 1,198 |
| BBY LN Equity | Balfour Beatty PLC | 1,160 |
| VSVS LN Equity | Vesuvius PLC | 1,138 |
| SNI NO Equity | Stolt-Nielsen Ltd | 1,109 |
| ASM NA Equity | ASM International NV | 1,066 |
| TW/ LN Equity | Taylor Wimpey PLC | 1,064 |
| IRV LN Equity | Interserve PLC | 943 |
| RIEN SE Equity | Rieter Holding AG | 941 |
| VCT FP Equity | Vicat SA | 920 |
| BOY LN Equity | Bodycote PLC | 905 |
| HUH1V FH Equity | Huhtamaki OYJ | 898 |
| MGGT LN Equity | Meggitt PLC | 893 |
| JYSK DC Equity | Jyske Bank A/S | 861 |
| WHA NA Equity | Wereldhave NV | 841 |
| GPOR LN Equity | Great Portland Estates PLC | 814 |
| MGAM LN Equity | Morgan Advanced Materials PLC | 727 |

*Source: Bloomberg L.P. (2019)*